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A self-consistent Eulerian rate type model for finite deformation elastoplasticity with isotropic damage

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Abstract

Continuum models for coupled behaviour of elastoplasticity and isotropic damage at finite deformation are usually formulated by first postulating the additive decomposition of the stretching tensor **D** into the elastic and the plastic part and then relating each part to an objective rate of the effective stress, etc. It is pointed out that, according to the existing models with several widely used objective stress rates, none of the rate equations intended for characterizing the damaged elastic response is exactly integrable to really deliver a damaged elastic relation between the effective stress and an elastic strain measure. The existing models are thus *self-inconsistent* in the sense of formulating the damaged elastic response. By consistently combining additive and multiplicative decomposition of the stretching **D** and the deformation gradient **F** and adopting the logarithmic stress rate, in this article, we propose a general Eulerian rate type model for finite deformation elastoplasticity coupled with isotropic damage. The new model is shown to be self-consistent in the sense that the incorporated rate equation for the damaged elastic response is exactly integrable to yield a damaged elastic relation between the effective Kirchhoff stress and the elastic logarithmic strain. The rate form of the new model in a rotating frame in which the foregoing logarithmic rate is defined, is derived and from it an integral form is obtained. The former is found to have the same structure as the counterpart of the small deformation theory and may be appropriate for numerical integration. The latter indicates, in a clear and direct manner, the effect of finite rotation and deformation history on the current stress and the hardening and damage behaviours. Further, it is pointed out that in the foregoing self-consistency sense of formulating the damaged elastic response, the suggested model is unique among all objective Eulerian rate type models of its kind with infinitely many objective stress rates to be chosen. In particular, it is indicated that, within the context of the proposed theory, a natural combination of the two widely used decompositions concerning **D** and **F** can consistently and uniquely determine the elastic and the plastic parts in the two decompositions as well as all their related kinematical quantities, without recourse to any ad hoc assumption concerning a special form of the elastic part $\mathbf{F}^e$ in the decomposition $\mathbf{F} = \mathbf{F}^e\mathbf{F}^p$ or a related relaxed intermediate configuration. As an application, the proposed general model is applied to derive a self-consistent Eulerian rate type model for void growth and nucleation in metals experiencing finite elastic–plastic deformation by incorporating a modified Gurson's yield function and an associated flow rule, etc. Two issues involved in previous relevant literature are detected and raised for consideration. As a test problem, the finite simple shear response of the just-mentioned model is studied by means of numerical integration.

Keywords: Finite elastoplasticity; Damage; Logarithmic rate

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1. Introduction

It is widely recognized that in a deforming material body, evolution of microstructure, such as microdefects, microvoids and microcracks, etc. is the main cause leading to irreversible inelastic deformations. On the other hand, deformation, in particular large deformation, usually causes changes of microstructure in a material body. The actual coupling mechanism between the process of deformation and the evolution of microstructure may be extremely complicated in nature. In an idealized and simplified sense, a macroscopic scalar variable ϕ called *damage variable*, among other things, may be introduced to represent the state of microstructure and is directly associated with pertinent mechanical quantities, such as the stress, the material moduli, etc. Then, a phenomenological model for the foregoing coupling mechanism may be established by formulating the evolution equation of the damage variable ϕ and other relevant rate type constitutive equations. Since the inception of the seminal idea by Kachanov (1958), the very promising branch of continuum mechanics, *continuum damage mechanics*, has been developing extensively and steadily and receiving increasing applications in numerous related fields, refer to, e.g. Kachanov (1986), Krajcinovic and Lemaitre (1987), Chaboche (1988), Lemaitre and Chaboche (1990), Lemaitre (1992), Krajcinovic (1996) and the relevant literature therein for details.

At the present stage of development, a set of damage variables and other internal variables of scalar type and tensorial type are introduced to characterize the state of microstructure of a material in a more realistic manner and more general models are accordingly developed, see the aforementioned monographs and recent works by, e.g., Onat and Leckie (1988), Bruhns and Diehl (1989), Voyiadjis and Kattan (1992a,b), Lubarda (1994), Lubarda and Krajcinovic (1995), and Bruhns and Schieße (1996), and others. This general aspect is still under continuing development. In this article, we are mainly concerned with the classical aspect, i.e. the *isotropic damage* with one scalar damage variable ϕ. This aspect has been fully studied with reference to both small and finite deformation due to its simple, clear and direct physical meaning. Now, it may be said that isotropic damage theories with reference to small deformation are well established on firm mathematical and physical foundations. However, the case might not be so when large deformation is concerned. In fact, even the existing formulations of finite deformation elastoplasticity are somewhat controversial and a number of fundamental issues between them have been indicated and extensively debated (see, e.g., the recent comprehensive review by Naghdi (1990) and the pertinent references therein for detail). As a result, finite deformation elastoplastic damage theories based on them are accordingly subject to the same issues.

Based on some recent developments in kinematics of finite deformation and rate type constitutive models by these authors and other researchers (see Bruhns et al., 1999; Xiao et al., 1996, 1997a,b, 1998a,b, 1999, 2000; Lehmann et al., 1991; Reinhardt and Dubey, 1995, 1996), we shall establish a general Eulerian rate type model for finite deformation elastoplasticity coupled with isotropic damage, with which the main fundamental discrepancies involved in existing formulations of finite elastoplasticity disappear. The main content of this article is arranged as follows: In Section 2, for later use we introduce the newly discovered logarithmic rate and the rotating frame in which the latter is defined, as well as other basic facts regarding kinematics of finite deformations of continua. In Section 3, postulating the additive decomposition of the stretching $\mathbf{D}$ and adopting the logarithmic rate, we establish a complete system of Eulerian rate type constitutive equations governing the coupled behaviour of finite elastoplasticity and isotropic damage. It is pointed out that the logarithmic rate is a unique choice among all infinitely many objective corotational rates, in the self-consistency sense of achieving an integrable-exactly rate type formulation of damaged elastic response. In Section 4, the elastic and the plastic part in the decomposition $\mathbf{F} = \mathbf{F}^e\mathbf{F}^p$ and all their related kinematical quantities are uniquely and consistently determined. In Section 5, we supply the rate form of the suggested constitutive formulation in a rotating frame in which the logarithmic rate is defined and then derive an integral formulation. Some implications of the results obtained are indicated. In Section 6, incorporating a modified Gurson's yield function and an associated flow rule, etc. we apply the general model proposed in Section 3 to derive a self-consistent Eulerian rate type model for void growth and nucleation in porous metals at finite deformation. Two issues involved in previous relevant literature are detected and raised for consideration. Finally, in Section 7, we study the finite simple shear response of the model established in Section 6 by means of numerical integration.

Let $\mathbf{Q}$, $\mathbf{A}$, $\mathbf{B}$ and $\mathbb{H}$ be, respectively, an orthogonal tensor, two second-order tensors and a fourth-order tensor. We shall use the notations $\mathbf{A}:\mathbf{B}$, $\mathbf{AB}$, $\mathbf{Q}*\mathbf{A}$, $\mathbb{H}:\mathbf{A}$ and $\mathbf{Q}*\mathbb{H}$ to designate, respectively, the scalar, the three second-order tensors and the fourth-order tensor given by

$$\mathbf{A}:\mathbf{B} = A_{ij}B_{ij},$$

$$(\mathbf{AB})_{ij} = A_{ik}B_{kj},$$

$$(\mathbf{Q}*\mathbf{A})_{ij} = Q_{ik}Q_{jl}A_{kl},$$

$$(\mathbb{H}:\mathbf{A})_{ij} = \mathbb{H}_{ijkl}A_{kl},$$

$$(\mathbf{Q}*\mathbb{H})_{ijkl} = Q_{ip}Q_{jq}Q_{kr}Q_{ls}\mathbb{H}_{pqrs}.$$

The following identities will be useful:

$$(\mathbf{Q}*\mathbf{A}):(\mathbf{Q}*\mathbf{B}) = \mathbf{A}:\mathbf{B},$$

$$(\mathbf{Q}*\mathbb{H}):(\mathbf{Q}*\mathbf{A}) = \mathbf{Q}*(\mathbb{H}:\mathbf{A}).$$

2. Logarithmic rate and logarithmic rotating frame

Consider a deforming body with particles. We identify each particle with a position vector $\mathbf{X}$ in a referential configuration, e.g. an initial configuration. The current position vector of a particle $\mathbf{X}$ is denoted by $\mathbf{x} = \bar{\mathbf{x}}(\mathbf{X}, t)$, and hence the velocity vector of a particle $\mathbf{X}$ is given by $\mathbf{v} = \dot{\mathbf{x}}$. Throughout, the superposed dot is used to represent the material time derivative.

The local deformation state at a particle $\mathbf{X}$ is described by the *deformation gradient*

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}},$$

while the rate of change of deformation state at a particle $\mathbf{X}$ is characterized by the *velocity gradient*

$$\mathbf{L} = \frac{\partial \mathbf{v}}{\partial \mathbf{x}} = \dot{\mathbf{F}}\mathbf{F}^{-1}.$$

The following left polar decomposition formula and additive decomposition formula are well known:

$$\mathbf{F} = \mathbf{VR}, \quad \mathbf{R}^{\mathrm{T}} = \mathbf{R}^{-1}, \quad \mathbf{B} = \mathbf{V}^2 = \mathbf{FF}^{\mathrm{T}},$$

$$\mathbf{L} = \mathbf{W} + \mathbf{D}, \quad \mathbf{W} = \tfrac{1}{2}(\mathbf{L} - \mathbf{L}^{\mathrm{T}}), \quad \mathbf{D} = \tfrac{1}{2}(\mathbf{L} + \mathbf{L}^{\mathrm{T}}).$$

The symmetric positive definite tensors $\mathbf{V}$ and $\mathbf{B}$ are known as, respectively, the *left stretch tensor* and the *left Cauchy–Green tensor*, the proper orthogonal tensor $\mathbf{R}$ is the *rotation tensor*, and the symmetric and antisymmetric tensors $\mathbf{D}$ and $\mathbf{W}$ are called the *stretching* and the *vorticity tensor*. Throughout, $\mathbf{S}^{\mathrm{T}}$ and $\mathbf{S}^{-1}$ are used to denote the transpose and the inverse of the second-order tensor $\mathbf{S}$.

Let the distinct eigenvalues of the left Cauchy–Green tensor $\mathbf{B}$ be given by $\chi_1, \ldots, \chi_m$ and their corresponding subordinate eigenprojections by $\mathbf{B}_1, \ldots, \mathbf{B}_m$. We introduce a general class of Eulerian strain measures by (see Hill, 1978; Ogden, 1984; see also Xiao et al., 1998a)

$$\mathbf{e} = \mathbf{g}(\mathbf{B}) = \sum_{\alpha=1}^{m} g(\chi_\alpha)\mathbf{B}_\alpha, \tag{1}$$

where g: $R^+ \to R$, called *scale function*, is a smooth monotonic increasing function with the normalized property $g(1) = 2g'(1) - 1 = 0$. Taking the scale function $g(\chi)$ as certain particular forms, one can obtain almost all commonly-known Eulerian strain measures. In particular, by taking $g(\chi) = \frac{1}{2}\ln \chi$, Hencky's Eulerian *logarithmic strain measure*

$$\mathbf{h} = \tfrac{1}{2}\ln \mathbf{B} = \tfrac{1}{2}\sum_{\alpha=1}^{m}(\ln \chi_\alpha)\mathbf{B}_\alpha \tag{2}$$

is available, which will be of particular interest.

On the other hand, let $\boldsymbol{\Omega}^*$ be a spin, i.e. a time-dependent antisymmetric second-order tensor. In a rotating frame with the spin $\boldsymbol{\Omega}^*$, an objective Eulerian symmetric second-order tensor $\mathbf{S}$ in a fixed background frame, becomes $\mathbf{QSQ}^{\mathrm{T}}$, and hence its time rate in this rotating frame is given by

$$\overline{(\mathbf{QSQ}^{\mathrm{T}})}^{\boldsymbol{\cdot}} = \mathbf{Q}\dot{\mathbf{S}}\mathbf{Q}^{\mathrm{T}} + \dot{\mathbf{Q}}\mathbf{S}\mathbf{Q}^{\mathrm{T}} + \mathbf{Q}\mathbf{S}\dot{\mathbf{Q}}^{\mathrm{T}} = \mathbf{Q}\overset{\circ}{\mathbf{S}}{}^{*}\mathbf{Q}^{\mathrm{T}}. \tag{3}$$

In the above, $\mathbf{Q}$ is a proper orthogonal tensor defining the spin $\boldsymbol{\Omega}^*$, i.e.

$$\boldsymbol{\Omega}^* = \dot{\mathbf{Q}}^{\mathrm{T}}\mathbf{Q} \tag{4}$$

and moreover

$$\overset{\circ}{\mathbf{S}}{}^{*} = \dot{\mathbf{S}} + \mathbf{S}\boldsymbol{\Omega}^* - \boldsymbol{\Omega}^*\mathbf{S}. \tag{5}$$

It follows from Eq. (3) that the latter, called the corotational rate of the tensor $\mathbf{S}$ defined by the spin $\boldsymbol{\Omega}^*$, is just the counterpart of the time rate of $\mathbf{QSQ}^{\mathrm{T}}$ in a background frame. It is evident that there are infinitely many kinds of corotational rates. Not all of them, however, are objective. A well-known example of objective corotational rate is provided by the Zaremba–Jaumann rate with $\boldsymbol{\Omega}^* = \mathbf{W}$, and another well-known example is given by the Green–Naghdi rate with the polar spin $\boldsymbol{\Omega}^* = \dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}}$. In general, the objectivity of a corotational rate depends on its defining spin tensor. The latter must be associated with the rotation and deformation of the deforming body in question, as is shown by several commonly known examples. A general class of objective corotational rates and their defining spin tensors have been derived by these authors (Xiao et al., 1998b).

It is commonly accepted that the stretching tensor $\mathbf{D}$, the symmetric part of the velocity gradient, is a well-defined fundamental kinematic quantity measuring the rate of change of local deformation state in a deforming body. It is frequently referred to as the Eulerian strain rate, the tensor of deformation rate, or simply the deformation rate. However, it has long been unknown whether or not the stretching can be really written as a rate of a strain measure. The pertinent question is: whether or not a strain measure $\mathbf{e}$ and a spin $\boldsymbol{\Omega}^*$ can be found such that the objective corotational rate of $\mathbf{e}$ defined by $\boldsymbol{\Omega}^*$ is exactly identical with the stretching tensor $\mathbf{D}$, i.e.

$$\overset{\circ}{\mathbf{e}}{}^{*} = \dot{\mathbf{e}} + \mathbf{e}\boldsymbol{\Omega}^* - \boldsymbol{\Omega}^*\mathbf{e} = \mathbf{D}. \tag{6}$$

It turns out (see Xiao et al., 1996, 1997a, 1998a) that the above expression, where both the strain measure $\mathbf{e}$ and the spin $\boldsymbol{\Omega}^*$ are left to be determined, holds iff the strain measure $\mathbf{e}$ is the logarithmic strain $\mathbf{h}$ given by Eq. (2), i.e.

$$g(\chi) = \tfrac{1}{2}\ln\chi. \tag{7}$$

When $\mathbf{e} = \mathbf{h}$, the linear tensor equation (Eq. (6)) has a unique continuous solution to the spin $\boldsymbol{\Omega}^*$, denoted by $\boldsymbol{\Omega}^{\log}$ and given by (see Xiao et al., 1996, 1997a, 1998a)

$$\boldsymbol{\Omega}^{\log} = \mathbf{W} + \sum_{\sigma\neq\tau}^{m}\left(\frac{1+(\chi_\sigma/\chi_\tau)}{1-(\chi_\sigma/\chi_\tau)} + \frac{2}{\ln(\chi_\sigma/\chi_\tau)}\right)\mathbf{B}_\sigma\mathbf{D}\mathbf{B}_\tau. \tag{8}$$

An explicit basis-free expression for $\boldsymbol{\Omega}^{\log}$ can be found in Xiao et al. (1996, 1997a, 1998a). Owing to the unique relationship between the stretching $\mathbf{D}$ and the logarithmic strain $\mathbf{h}$ indicated, the spin $\boldsymbol{\Omega}^{\log}$ has been termed the *logarithmic spin* and accordingly the objective corotational rate defined by it the *logarithmic rate*.

The logarithmic rate of an Eulerian symmetric second-order tensor $\mathbf{S}$ is denoted by $\overset{\circ}{\mathbf{S}}{}^{\log}$, i.e.

$$\overset{\circ}{\mathbf{S}}{}^{\log} \equiv \dot{\mathbf{S}} + \mathbf{S}\boldsymbol{\Omega}^{\log} - \boldsymbol{\Omega}^{\log}\mathbf{S}. \tag{9}$$

In particular, we have

$$\overset{\circ}{\mathbf{h}}{}^{\log} = \mathbf{D}. \tag{10}$$

Let $\mathbf{R}^{\log}$ be the proper orthogonal tensor defining the logarithmic spin $\boldsymbol{\Omega}^{\log}$, which is called the *logarithmic rotation* and determined by the tensor differential equation

$$\dot{\overline{\mathbf{R}^{\log}}} = -\mathbf{R}^{\log}\boldsymbol{\Omega}^{\log} \tag{11}$$

with the initial condition

$$\mathbf{R}^{\log}\,|_{t=0} = \mathbf{I}. \tag{12}$$

Then we have (cf. Eqs. (3) and (4))

$$\dot{\overline{(\mathbf{R}^{\log} * \mathbf{S})}} = \mathbf{R}^{\log} * \overset{\circ}{\mathbf{S}}{}^{\log} \tag{13}$$

for any Eulerian symmetric second-order tensor $\mathbf{S}$. In particular, by setting $\mathbf{S} = \mathbf{h}$ in the above and using Eq. (10), we have

$$\dot{\overline{(\mathbf{R}^{\log} * \mathbf{h})}} = \mathbf{R}^{\log} * \mathbf{D}. \tag{14}$$

The logarithmic rotation $\mathbf{R}^{\log}$ defines a rotating frame via the transformation of motion

$$\bar{\mathbf{x}}^+(\mathbf{X},t) = \mathbf{x}_0(t) + \mathbf{R}^{\log}\bar{\mathbf{x}}(\mathbf{X},t). \tag{15}$$

This frame, whose spin is just the logarithmic spin $\boldsymbol{\Omega}^{\log}$ due to Eq. (11), is called the logarithmic rotating frame. The equality (Eq. (14)) indicates a kinematical feature of the logarithmic rotation or the logarithmic spin: An observer in the logarithmic rotating frame observes that the material time derivative of Hencky's logarithmic strain measure is just the stretching.

Thus, the logarithmic rotation is associated with the deformation and rotation in a deforming body in a unique manner. It is evident that such an association is purely of kinematical character and independent of any material behaviour.

Lehmann et al. (1991) were the first to consider the particular case of the tensor equation (Eq. (6)) when $\mathbf{e} = \mathbf{h}$ and introduce the logarithmic spin $\boldsymbol{\Omega}^{\log}$. Similar results were derived later by Reinhardt and Dubey (1995, 1996). In a different context, these authors (Xiao et al., 1996, 1997a, 1998a) studied the general case of tensor equation (Eq. (6)) with both the strain measure $\mathbf{e}$ and the spin $\boldsymbol{\Omega}^*$ left to be determined and revealed the unique relationship between the stretching $\mathbf{D}$ and the logarithmic strain $\mathbf{h}$ for the first time. The significance of the logarithmic rate to formulating Eulerian rate type inelasticity models has been indicated in very recent works by these authors (Bruhns et al., 1999; Xiao et al., 1997a,b, 1999, 2000). Based on these results, in the succeeding sections we shall develop new Eulerian rate type models for finite deformation elastoplasticity coupled with isotropic damage.

3. Eulerian rate type constitutive formulation

We consider a damaged elastoplastic solid with an initial stress-free natural state $\mathscr{C}_0$ and with initial isotropy material symmetry. The initial natural state is taken as the reference configuration. Accordingly, we have the initial conditions

$$\mathbf{F}\,|_{t=0} = \mathbf{I}, \qquad \boldsymbol{\tau}\,|_{t=0} = \mathbf{O}. \tag{16a, b}$$

Here and henceforth, $\boldsymbol{\tau}$ is used to denote the *Kirchhoff stress*, which is related to the Cauchy stress $\boldsymbol{\sigma}$ by

$$\boldsymbol{\tau} = (\det \mathbf{F})\boldsymbol{\sigma}.$$

We assume that the damage variable is a scalar variable ϕ whose value belongs to the interval $[0, 1]$. Then we define the *effective Kirchhoff stress* as follows:

$$\bar{\boldsymbol{\tau}} = \frac{\boldsymbol{\tau}}{1-\phi}. \tag{17}$$

As commonly done (see, e.g., Nemat-Nasser, 1979, 1982), we assume the additive decomposition of the stretching $\mathbf{D}$:

$$\mathbf{D} = \mathbf{D}^{\mathrm{e}} + \mathbf{D}^{\mathrm{ep}}. \tag{18}$$

We call $\mathbf{D}^{\mathrm{e}}$ and $\mathbf{D}^{\mathrm{ep}}$ the elastic part and the coupled elastic–plastic part of $\mathbf{D}$. Another widely used decomposition is the multiplicative decomposition of the deformation gradient $\mathbf{F}$ (see Eq. (38) given later). A natural and consistent combination of the two kinds of widely used decompositions will be given in the Section 4. It will be seen that the elastic part $\mathbf{D}^{\mathrm{e}}$ of $\mathbf{D}$ is related to the elastic part $\mathbf{F}^{\mathrm{e}}$ of $\mathbf{F}$ (see Eq. (42a), while $\mathbf{D}^{\mathrm{ep}}$ is associated with both the elastic part $\mathbf{F}^{\mathrm{e}}$ and the plastic part $\mathbf{F}^{\mathrm{p}}$ of $\mathbf{F}$ (see Eq. (42b)).

In the succeeding subsections we shall establish Eulerian rate type constitutive formulations for the two parts $\mathbf{D}^{\mathrm{e}}$ and $\mathbf{D}^{\mathrm{ep}}$ as well as the damage variable ϕ etc., respectively.

3.1. Integrable-exactly Eulerian rate type formulation of general damaged hyperelasticity

The elastic part $\mathbf{D}^{\mathrm{e}}$ is related to an objective rate, $\overset{\circ}{\bar{\boldsymbol{\tau}}}{}^{*}$, of the effective Kirchhoff stress $\bar{\boldsymbol{\tau}}$ in a form

$$\mathbf{D}^{\mathrm{e}} = \mathbb{D} : \overset{\circ}{\bar{\boldsymbol{\tau}}}{}^{*} \tag{19}$$

or

$$\overset{\circ}{\bar{\boldsymbol{\tau}}}{}^{*} = \mathbb{C} : \mathbf{D}^{\mathrm{e}}. \tag{20}$$

In the above, the fourth-order tensor $\mathbb{D}$ or its inverse $\mathbb{D}^{-1} = \mathbb{C}$ characterize the instantaneous elastic behaviour of the material. Most often $\mathbb{D}$ is chosen as the constant isotropic compliance tensor, especially for small elastic strain case. Hence,

$$\mathbb{D} = \left(\frac{1}{E} - \frac{1}{2G}\right)\mathbf{I}\otimes\mathbf{I} + \frac{1}{2G}\mathbb{I}, \tag{21}$$

where E and G are Young's modulus and the shear modulus, respectively. Throughout, $\mathbf{I}$ and $\mathbb{I}$ are used to designate the second-order and the fourth-order symmetric identity tensor, respectively, i.e.

$$(\mathbf{I})_{ij} = \delta_{ij}, \qquad (\mathbb{I})_{ijkl} = \tfrac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{22}$$

with the Kronecker delta δ_{rs}.

In the rate equations (Eqs. (19) or (20)), the choice of the objective rate $\overset{\circ}{\bar{\boldsymbol{\tau}}}{}^{*}$ is crucial. It should be noted that these equations are intended to characterize damaged elastic response. Namely, they must be exactly integrable to really deliver a damaged elastic, in particular hyperelastic, relation. However, if special care is not taken, the just-mentioned self-consistency requirement for rate-type characterization of damaged elastic response may not be fulfilled and some aberrant, spurious phenomena, such as the oscillatory shear stress response with increasing shearing strain, etc., may be resulted in, as disclosed by Lehmann (1972a,b), Dienes (1979) and Nagtegaal and de Jong (1982), and others, for the case of hypoelasticity and elastoplasticity without damage. Further, Simó and Pister (1984) have proved that none of the rate equations (Eqs. (19) or (20)) with several commonly known stress rates, such as Zaremba–Jaumann rate, Truesdell rate and Green–Naghdi rate etc., is integrable to yield an elastic, in particular hyperelastic, relation in nonlinear range. This fact indicates that the existing formulations of Eulerian rate type elastoplasticity (and accordingly elastoplasticity coupled with damage) are self-inconsistent in the sense of characterizing elastic response.

The undesirable self-inconsistency indicated above has been removed in very recent works by these authors for the case of hypoelasticity and elastoplasticity (see Bruhns et al., 1999; Xiao et al., 1997b, 1999). Utilizing these results, we here propose the integrable-exactly Eulerian rate type formulation of general damaged hyperelasticity as follows:

$$\mathbf{D}^{\mathrm{e}} = \frac{\partial^2 \Sigma}{\partial \bar{\boldsymbol{\tau}}\, \partial \bar{\boldsymbol{\tau}}} : \overset{\circ}{\bar{\boldsymbol{\tau}}}{}^{\log}, \tag{23}$$

where $\Sigma = \hat{\Sigma}(\bar{\boldsymbol{\tau}})$, which is an isotropic scalar function of the effective Kirchhoff stress $\bar{\boldsymbol{\tau}}$, is called the *effective complementary hyperelastic potential*. For small elastic strain, the gradient $\partial^2\Sigma/\partial\bar{\boldsymbol{\tau}}\,\partial\bar{\boldsymbol{\tau}}$ may be taken as the constant isotropic compliance tensor as shown by Eq. (21).

It will be shown in the next two sections that the rate equation (Eq. (23)) provides a consistent definition for the elastic deformation rate $\mathbf{D}^{\mathrm{e}}$ and leads to a general damaged hyperelastic relation according to which the elastic logarithmic strain measure $\mathbf{h}^{\mathrm{e}}$ (see Eq. (43)) is derivable from the potential $\hat{\Sigma}(\bar{\boldsymbol{\tau}})$ with respect to the effective Kirchhoff stress $\bar{\boldsymbol{\tau}}$, as will be shown by Eq. (44). Hence, defining a scalar function Σ' of $\mathbf{h}^{\mathrm{e}}$ via the Legendre transformation relation

$$\Sigma' + \Sigma = \bar{\boldsymbol{\tau}} : \mathbf{h}^{\mathrm{e}}$$

and using the equality (see Eq. (44) given later)

$$\dot{\Sigma} = \frac{\partial \Sigma}{\partial \bar{\boldsymbol{\tau}}} : \dot{\bar{\boldsymbol{\tau}}} = \mathbf{h}^{\mathrm{e}} : \dot{\bar{\boldsymbol{\tau}}},$$

we deduce

$$\dot{\Sigma}' = \overline{(\bar{\boldsymbol{\tau}} : \mathbf{h}^{\mathrm{e}})}^{\boldsymbol{\cdot}} - \dot{\Sigma} = \bar{\boldsymbol{\tau}} : \dot{\overline{\mathbf{h}^{\mathrm{e}}}}.$$

Then, from the latter and $\dot{\Sigma}' = \frac{\partial \Sigma'}{\partial \mathbf{h}^{\mathrm{e}}} : \dot{\overline{\mathbf{h}^{\mathrm{e}}}}$, we infer

$$\bar{\boldsymbol{\tau}} = \frac{\partial \Sigma'}{\partial \mathbf{h}^{\mathrm{e}}}.$$

In addition, noting that $\mathbf{h}^{\mathrm{e}}$ is given by an isotropic function (see Eq. (44)) of $\bar{\boldsymbol{\tau}}$ and hence that

$$\bar{\boldsymbol{\tau}} : (\mathbf{h}^{\mathrm{e}}\boldsymbol{\Omega}^{\log} - \boldsymbol{\Omega}^{\log}\mathbf{h}^{\mathrm{e}}) = 2(\mathbf{h}^{\mathrm{e}}\bar{\boldsymbol{\tau}}) : \boldsymbol{\Omega}^{\log} = 0,$$

and using the relationship (Eq. (45)), we have

$$\dot{\Sigma}' = \bar{\boldsymbol{\tau}} : \dot{\overline{\mathbf{h}^{e}}} = \bar{\boldsymbol{\tau}} : \overset{\circ}{\overline{\mathbf{h}^{e}}}{}^{\log} = \bar{\boldsymbol{\tau}} : \mathbf{D}^{e}.$$

In the above, the last expression implies that the material time derivative of the scalar function Σ' furnishes the effective elastic stress power, whereas the equation relating $\bar{\boldsymbol{\tau}}$ and $\mathbf{h}^{e}$ shows that the effective Kirchhoff stress $\bar{\boldsymbol{\tau}}$ is derivable from the scalar function Σ' with respect to the elastic logarithmic strain measure $\mathbf{h}^{e}$. These facts explain why the scalar function Σ in Eq. (23) has been termed the effective complementary hyperelastic potential.

The logarithmic stress rate used in the rate equation (23) is merely a particular objective stress rate among infinitely many objective corotational stress rates (see Xiao et al., 1998a,b). Probably another objective corotational stress rate may serve our purpose just as well. Hence, by replacing the logarithmic stress rate $\overset{\circ}{\bar{\boldsymbol{\tau}}}{}^{\log}$ with another objective stress rate $\overset{\circ}{\bar{\boldsymbol{\tau}}}{}^{*}$ one obtains another form of rate type equation for the elastic part $\mathbf{D}^{e}$ of $\mathbf{D}$

$$\mathbf{D}^{e} = \frac{\partial^2 \Sigma}{\partial \bar{\boldsymbol{\tau}}\, \partial \bar{\boldsymbol{\tau}}} : \overset{\circ}{\bar{\boldsymbol{\tau}}}{}^{*}.$$

The above-mentioned nonuniqueness, if any, will result in the puzzling situation concerning which stress rate is better, as encountered in existing Eulerian rate type formulations of finite elastoplasticity (see Khan and Huang (1995) for detail). Recently, these authors have demonstrated (see Bruhns et al., 1999; Xiao et al., 1999) that the above rate equation is exactly integrable to deliver an elastic relation if and only if the stress rate $\overset{\circ}{\bar{\boldsymbol{\tau}}}{}^{*}$ is the logarithmic rate $\overset{\circ}{\bar{\boldsymbol{\tau}}}{}^{\log}$, i.e. the rate equation is identical with the rate equation (Eq. (23)). This fact means that the rate equation (Eq. (23)) is unique among all the rate equations of its kind with infinitely many objective corotational rates to be chosen, in the self-consistency sense of formulating damaged elastic response.

3.2. *Yield function, flow potential and flow rules*

In addition to the damage variable ϕ, we introduce a scalar k and an objective symmetric second-order Eulerian tensor $\boldsymbol{\alpha}$ as internal variables to characterize isotropic and kinematic hardening behaviours. The tensor $\boldsymbol{\alpha}$ is called the *back* or *shift stress*. We assume that the current yield surface in the stress space is defined by

$$f = \hat{f}(\boldsymbol{\tau}, \boldsymbol{\alpha}, k, \phi) = 0. \tag{24}$$

Here $\hat{f}$ is an isotropic scalar function of the Kirchhoff stress and the back stress $\boldsymbol{\alpha}$. We further assume that in the stress space there is another surface

$$g = \hat{g}(\boldsymbol{\tau}, \boldsymbol{\alpha}, k, \phi), \tag{25}$$

such that the elastic–plastic part $\mathbf{D}^{ep}$ of the stretching $\mathbf{D}$ is in the direction of the gradient of this surface with respect to the Kirchhoff stress $\boldsymbol{\tau}$, i.e.

$$\mathbf{D}^{ep} = \dot{\lambda} \frac{\partial g}{\partial \boldsymbol{\tau}}. \tag{26}$$

Accordingly, g is called the *flow potential*, which is here also an isotropic scalar function of the Kirchhoff stress and the back stress.

For a process of continued plastic flow, the stress point must remain on the current yield surface. Hence, we have the consistency condition $\dot{f} = 0$ for plastic flow. Since the yield function f is isotropic, we have

$$\hat{f}(\boldsymbol{\tau}, \boldsymbol{\alpha}, k, \phi) = \hat{f}(\mathbf{R}^{\log} * \boldsymbol{\tau}, \mathbf{R}^{\log} * \boldsymbol{\alpha}, k, \phi).$$

Hence, we may write the just-mentioned condition in the form

$$\overline{\hat{f}(\mathbf{R}^{\log} * \boldsymbol{\tau}, \mathbf{R}^{\log} * \boldsymbol{\alpha}, k, \phi)}^{\bullet} = 0,$$

i.e.

$$\frac{\partial f^+}{\partial \boldsymbol{\tau}^+} : \dot{\overline{\boldsymbol{\tau}^+}} + \frac{\partial f^+}{\partial \boldsymbol{\alpha}^+} : \dot{\overline{\boldsymbol{\alpha}^+}} + \frac{\partial f^+}{\partial k}\dot{k} + \frac{\partial f^+}{\partial \phi}\dot{\phi} = 0,$$

where f^+ and $\boldsymbol{\tau}^+$ and $\boldsymbol{\alpha}^+$ are, respectively, the counterparts of the yield function f and the Kirchhoff stress $\boldsymbol{\tau}$ and the back stress $\boldsymbol{\alpha}$ in the logarithmic rotating frame, given by Eqs. (59a) and (58) later. Hence, utilizing the equality (Eq. (13)) and the equality following Eq. (67), as well as the penultimate identity in Section 1, we formulate the consistency condition for plastic flow in a form convenient for later use:

$$\frac{\partial f}{\partial \boldsymbol{\tau}} : \overset{\circ}{\boldsymbol{\tau}}{}^{\log} + \frac{\partial f}{\partial \boldsymbol{\alpha}} : \overset{\circ}{\boldsymbol{\alpha}}{}^{\log} + \frac{\partial f}{\partial k}\dot{k} + \frac{\partial f}{\partial \boldsymbol{\phi}}\dot{\phi} = 0. \tag{27}$$

Moreover, we assume general forms of evolution equations for the damage variable ϕ, the isotropic hardening parameter k and the back stress $\boldsymbol{\alpha}$ for kinematic hardening as follows:

$$\dot{\phi} = \boldsymbol{\phi} : \mathbf{D}^{\mathrm{ep}}, \quad \dot{k} = \mathbf{k} : \mathbf{D}^{\mathrm{ep}}, \quad \overset{\circ}{\boldsymbol{\alpha}}{}^{\log} = \mathbb{H} : \mathbf{D}^{\mathrm{ep}},$$

i.e.

$$\dot{\phi} = \dot{\lambda}\boldsymbol{\phi} : \frac{\partial g}{\partial \boldsymbol{\tau}}, \tag{28}$$

$$\dot{k} = \dot{\lambda}\mathbf{k} : \frac{\partial g}{\partial \boldsymbol{\tau}}, \tag{29}$$

$$\overset{\circ}{\boldsymbol{\alpha}}{}^{\log} = \dot{\lambda}\mathbb{H} : \frac{\partial g}{\partial \boldsymbol{\tau}}. \tag{30}$$

Here, the objective symmetric second-order Eulerian tensors $\boldsymbol{\phi}$ and $\mathbf{k}$ and fourth-order Eulerian tensor $\mathbb{H}$ depend on $\boldsymbol{\tau}$, $\boldsymbol{\alpha}$, k and ϕ, i.e.

$$\boldsymbol{\phi} = \hat{\boldsymbol{\phi}}(\boldsymbol{\tau}, \boldsymbol{\alpha}, k, \phi), \quad \mathbf{k} = \hat{\mathbf{k}}(\boldsymbol{\tau}, \boldsymbol{\alpha}, k, \phi), \quad \mathbb{H} = \hat{\mathbb{H}}(\boldsymbol{\tau}, \boldsymbol{\alpha}, k, \phi), \tag{31}$$

and each tensor-valued function above is isotropic with respect to $\boldsymbol{\tau}$ and $\boldsymbol{\alpha}$. In addition, the fourth-order tensor has minor index symmetry. Namely,

$$\hat{\boldsymbol{\phi}}(\mathbf{Q} * \boldsymbol{\tau}, \mathbf{Q} * \boldsymbol{\alpha}, k, \phi) = \mathbf{Q} * (\hat{\boldsymbol{\phi}}(\boldsymbol{\tau}, \boldsymbol{\alpha}, k, \phi)), \tag{32}$$

$$\hat{\mathbf{k}}(\mathbf{Q} * \boldsymbol{\tau}, \mathbf{Q} * \boldsymbol{\alpha}, k, \phi) = \mathbf{Q} * (\hat{\mathbf{k}}(\boldsymbol{\tau}, \boldsymbol{\alpha}, k, \phi)), \tag{33}$$

$$\begin{cases} \mathbb{H}_{ijkl} = \mathbb{H}_{jikl} = \mathbb{H}_{ijlk}, \\ \hat{\mathbb{H}}(\mathbf{Q} * \boldsymbol{\tau}, \mathbf{Q} * \boldsymbol{\alpha}, k, \phi) = \mathbf{Q} * (\hat{\mathbb{H}}(\boldsymbol{\tau}, \boldsymbol{\alpha}, k, \phi)), \end{cases} \tag{34}$$

for every orthogonal tensor $\mathbf{Q}$.

Utilizing Eqs. (28)–(30), from the consistency condition (Eq. (27)) for plastic flow, we derive an expression for the plastic multiplier $\dot{\lambda}$ as follows:

$$\begin{cases} \dot{\lambda} = -\frac{\psi}{\beta}\frac{\partial f}{\partial \boldsymbol{\tau}} : \overset{\circ}{\boldsymbol{\tau}}{}^{\log}, \\ \beta = \frac{\partial f}{\partial \boldsymbol{\alpha}} : \mathbf{H} : \frac{\partial g}{\partial \boldsymbol{\tau}} + \frac{\partial f}{\partial k}\left(\mathbf{k} : \frac{\partial g}{\partial \boldsymbol{\tau}}\right) + \frac{\partial f}{\partial \boldsymbol{\phi}}\left(\boldsymbol{\phi} : \frac{\partial g}{\partial \boldsymbol{\tau}}\right), \end{cases} \tag{35}$$

where the loading–unloading indicator ψ is of the form (see Bruhns et al., 1999)

$$\psi = \begin{cases} 0 & \text{if } f < 0 \text{ or } f = 0 \text{ and } \dfrac{\partial \hat{f}}{\partial \boldsymbol{\tau}} : \overset{\circ}{\boldsymbol{\tau}}^{\log} < 0, \\ 1 & \text{if } f = 0 \text{ and } \dfrac{\partial \hat{f}}{\partial \boldsymbol{\tau}} : \overset{\circ}{\boldsymbol{\tau}}^{\log} \geqslant 0. \end{cases} \tag{36}$$

Combining Eqs. (23) and (26) and (18), we arrive at

$$\mathbf{D} = \frac{\partial^2 \Sigma}{\partial \bar{\boldsymbol{\tau}} \partial \bar{\boldsymbol{\tau}}} : \overset{\circ}{\bar{\boldsymbol{\tau}}}^{\log} + \dot{\lambda} \frac{\partial g}{\partial \boldsymbol{\tau}}. \tag{37}$$

The Eulerian rate type constitutive equations (28)–(30) and (35)–(37), together with Cauchy's equations of motion, constitute a complete system of equations governing the total kinematical quantities and the total stress, etc. Of them, the basic elements are the complementary hyperelastic potential Σ, the yield function f, the flow potential g and the constitutive tensors $\boldsymbol{\phi}$, $\mathbf{k}$ and $\mathbb{H}$. For various kinds of materials, the latter may assume various forms. They must be determined by related experimental data. For example, various forms of evolution equations for the damage variable ϕ are available in Lemaitre and Chaboche (1990). This aspect can be simplified by using a yield function of von Mises type and the associated flow rule as well as simple hardening relations, as will be done in Section 6.

4. Kinematical quantities related to the decomposition $\mathbf{F} = \mathbf{F}^e\mathbf{F}^p$

The constitutive formulation proposed in Section 6, together with Cauchy's equations of motion and well-posed initial and boundary conditions, determine the total stress, the total kinematical quantities $\mathbf{F}$ and $\mathbf{L}$, as well as the elastic part $\mathbf{D}^e$ and the coupled elastic–plastic part $\mathbf{D}^{ep}$, etc. On the other hand, for a process of elastoplastic deformation, it is required to define and specify elastic and plastic deformations and their related kinematical quantities. It should be noted that if there is no a priori definition for elastic and plastic deformations, no definite information about the latter can be drawn from the rate quantities $\mathbf{D}^e$ and $\mathbf{D}^{ep}$. On the contrary, $\mathbf{D}^e$ and $\mathbf{D}^{ep}$ need to be related to "elastic deformation" and "plastic deformation" in an appropriate sense.

To introduce and separate elastic and plastic deformations, the physically motivated multiplicative decomposition of the total deformation gradient is widely used, which was first introduced by Kröner (1960) with reference to a linearized theory, subsequently utilized by Backmann (1964) and Willis (1969), and systematically and extensively used and developed by Lee and other researchers, see, e.g., Lee and Liu (1967), Lee (1969); see also, e.g., Lubarda (1994) and Lubarda and Krajcinovic (1995) for recent applications in continuum damage mechanics. According to this decomposition, the total deformation gradient $\mathbf{F}$ has the multiplicative decomposition

$$\mathbf{F} = \mathbf{F}^e\mathbf{F}^p, \quad \det \mathbf{F}^e > 0, \quad \det \mathbf{F}^p > 0, \tag{38}$$

for any process of elastic–plastic deformation. Usually, $\mathbf{F}^e$ and $\mathbf{F}^p$ are called, respectively, the elastic and the plastic part of $\mathbf{F}$.

The decomposition (38) produces

$$\mathbf{L} = \dot{\mathbf{F}}^e\mathbf{F}^{e-1} + \mathbf{F}^e\dot{\mathbf{F}}^p\mathbf{F}^{p-1}\mathbf{F}^{e-1}, \tag{39}$$

$$\mathbf{D} = \operatorname{sym}(\dot{\mathbf{F}}^e\mathbf{F}^{e-1}) + \operatorname{sym}(\mathbf{F}^e\dot{\mathbf{F}}^p\mathbf{F}^{p-1}\mathbf{F}^{e-1}), \tag{40}$$

$$\mathbf{W} = \operatorname{skw}(\dot{\mathbf{F}}^e\mathbf{F}^{e-1}) + \operatorname{skw}(\mathbf{F}^e\dot{\mathbf{F}}^p\mathbf{F}^{p-1}\mathbf{F}^{e-1}). \tag{41}$$

Here and henceforth we use the notations $\operatorname{sym}\mathbf{A}$ and $\operatorname{skw}\mathbf{A}$ to designate the symmetric and the skew-symmetric part of a second-order tensor $\mathbf{A}$, i.e.

$$\mathrm{sym}\,\mathbf{A} = \tfrac{1}{2}(\mathbf{A} + \mathbf{A}^{\mathrm{T}}), \qquad \mathrm{skw}\,\mathbf{A} = \tfrac{1}{2}(\mathbf{A} - \mathbf{A}^{\mathrm{T}}).$$

Now, we proceed to establish the relationship between the two decompositions (18) and (38). Towards this goal, let us compare Eqs. (18) and (40). Clearly, the first term of the right-hand side of Eq. (40) relies on the elastic part $\mathbf{F}^{\mathrm{e}}$ only, whereas the second term depends on both the elastic and the plastic part $\mathbf{F}^{\mathrm{e}}$ and $\mathbf{F}^{\mathrm{p}}$. Thus, a natural, direct relationship between the two decompositions (18) and (38) should be

$$\mathbf{D}^{\mathrm{e}} = \mathrm{sym}(\dot{\mathbf{F}}^{\mathrm{e}}\mathbf{F}^{\mathrm{e}-1}), \qquad \mathbf{D}^{\mathrm{ep}} = \mathrm{sym}(\mathbf{F}^{\mathrm{e}}\dot{\mathbf{F}}^{\mathrm{p}}\mathbf{F}^{\mathrm{p}-1}\mathbf{F}^{\mathrm{e}-1}). \tag{42a, b}$$

In the above two relations, the former implies the latter and vice versa. The right-hand side of the latter explains why $\mathbf{D}^{\mathrm{ep}}$ has been termed the coupled elastic–plastic part of $\mathbf{D}$ before.

Consider the constitutive formulation (23) for the elastic part $\mathbf{D}^{\mathrm{e}}$ of the stretching $\mathbf{D}$. As pointed out before, the rate equation (23) should be exactly integrable to produce a damaged elastic relation. Define the *elastic logarithmic strain measure* $\mathbf{h}^{\mathrm{e}}$ by

$$\mathbf{h}^{\mathrm{e}} = \tfrac{1}{2}\ln(\mathbf{F}^{\mathrm{e}}\mathbf{F}^{\mathrm{eT}}). \tag{43}$$

Generally, we may assume that the foregoing damaged elastic relation is of the form

$$\boldsymbol{\psi}(\mathbf{h}^{\mathrm{e}}) = \frac{\partial \Sigma}{\partial \bar{\boldsymbol{\tau}}}.$$

Now, we prove that $\boldsymbol{\psi}(\mathbf{h}^{\mathrm{e}})$ is exactly $\mathbf{h}^{\mathrm{e}}$. In fact, for each process of purely elastic deformation, i.e., $\mathbf{F}^{\mathrm{e}} = \mathbf{F}$, we have $\mathbf{h}^{\mathrm{e}} = \mathbf{h}$ and $\mathbf{D}^{\mathrm{e}} = \mathbf{D}$. Then, Eq. (23) and the foregoing relation become

$$\mathbf{D} = \frac{\partial^2 \Sigma}{\partial \bar{\boldsymbol{\tau}}\,\partial \bar{\boldsymbol{\tau}}} : \overset{\circ}{\bar{\boldsymbol{\tau}}}{}^{\log}, \qquad \boldsymbol{\psi}(\mathbf{h}) = \frac{\partial \Sigma}{\partial \bar{\boldsymbol{\tau}}}.$$

Hence we infer

$$\overset{\circ}{\overline{\boldsymbol{\psi}(\mathbf{h})}}{}^{\log} = \mathbf{D}.$$

Since the potential $\Sigma = \hat{\Sigma}(\bar{\boldsymbol{\tau}})$ is isotropic, the gradient $\partial\Sigma/\partial\bar{\boldsymbol{\tau}}$ and hence $\boldsymbol{\psi}(\mathbf{h})$ are also isotropic. Applying the chain rule for the gradient of a symmetric second-order tensor-valued isotropic function derived in Xiao et al. (1999), from the last equality we deduce

$$\frac{\partial \boldsymbol{\psi}}{\partial \mathbf{h}} : \overset{\circ}{\mathbf{h}}{}^{\log} = \mathbf{D}.$$

Thus, from the latter and the formula (Eq. (10)), we derive

$$\frac{\partial \boldsymbol{\psi}}{\partial \mathbf{h}} = \mathbf{I},$$

i.e., $\boldsymbol{\psi}(\mathbf{h}) = \mathbf{h}$ and hence $\boldsymbol{\psi}(\mathbf{h}^{\mathrm{e}}) = \mathbf{h}^{\mathrm{e}}$. From the above account, it follows that the elastic relation assumed before must take the form

$$\mathbf{h}^{\mathrm{e}} = \frac{\partial \Sigma}{\partial \bar{\boldsymbol{\tau}}}. \tag{44}$$

This and the rate equation (23) result in the relationship

$$\mathbf{D}^{\mathrm{e}} = \overset{\circ}{\overline{\mathbf{h}^{\mathrm{e}}}}{}^{\log}. \tag{45}$$

This means that the elastic part $\mathbf{D}^{\mathrm{e}}$ of the total stretching $\mathbf{D}$ is just the logarithmic rate of the elastic logarithmic strain measure $\mathbf{h}^{\mathrm{e}}$. Further interpretation of this relationship will be given in Section 5.

Furthermore, as has been indicated in Xiao et al. (2000), the above established relationship between the two widely used decompositions (18) and (38) can consistently and uniquely determine the elastic part $\mathbf{F}^e$ and the plastic part $\mathbf{F}^p$ in the decomposition (38), as well as all their related kinematical quantities, with no ad hoc assumption about restricted special forms of $\mathbf{F}^e$ and/or $\mathbf{F}^p$. Indeed, from the outset of this section we know that the effective Kirchhoff stress $\bar{\boldsymbol{\tau}}$, the total deformation gradient $\mathbf{F}$, the total velocity gradient $\mathbf{L}$, and the two parts $\mathbf{D}^e$ and $\mathbf{D}^{ep}$ of $\mathbf{D}$, can be obtained by integrating the constitutive equations (28)–(30), (35)–(37) and Cauchy equations of motion with well-posed initial and boundary conditions. Then, the elastic deformation $\mathbf{F}^e = \mathbf{V}^e\mathbf{R}^e$ over a time interval $[0, a]$ is consistently and uniquely determined by $\mathbf{V}^e$ and $\mathbf{D}^e$ given over $[0, a]$, where the elastic stretch tensor $\mathbf{V}^e = \sqrt{\mathbf{F}^e\mathbf{F}^{eT}}$ is determined by (see Eq. (44))

$$\mathbf{V}^e = \exp\left(\frac{\partial \Sigma}{\partial \bar{\boldsymbol{\tau}}}\right), \tag{46}$$

and the *elastic rotation* $\mathbf{R}^e$ is obtained by integrating the linear tensorial differential equation (see Eq. (61) in Xiao et al. (2000))

$$\dot{\mathbf{R}}^e = \boldsymbol{\Omega}^e\mathbf{R}^e, \qquad \mathbf{R}^e|_{t=0} = \mathbf{I}, \tag{47}$$

with (see Eq. (64) in Xiao et al. (2000))

$$\boldsymbol{\Omega}^e = \boldsymbol{\Omega}^{\log} - \sum_{\sigma\neq\tau}^{m}\left(\frac{2\lambda_\sigma^e\lambda_\tau^e}{\lambda_\tau^{e2} - \lambda_\sigma^{e2}} + \frac{1}{\ln\lambda_\sigma^e - \ln\lambda_\tau^e}\right)\mathbf{V}_\sigma^e\mathbf{D}^e\mathbf{V}_\tau^e. \tag{48}$$

In addition, we have

$$\dot{\mathbf{V}}^e = \boldsymbol{\Omega}^{\log}\mathbf{V}^e - \mathbf{V}^e\boldsymbol{\Omega}^{\log} + \sum_{\sigma,\tau=1}^{m}\frac{\lambda_\sigma^e - \lambda_\tau^e}{\ln\lambda_\sigma^e - \ln\lambda_\tau^e}\mathbf{V}_\sigma^e\mathbf{D}^e\mathbf{V}_\tau^e. \tag{49}$$

In the above, λ_σ^e and $\mathbf{V}_\sigma^e$, $\sigma = 1, \ldots, m$, are the distinct eigenvalues of the elastic stretch $\mathbf{V}^e$ and the corresponding subordinate *eigenprojections* of $\mathbf{V}^e$, respectively.

Once the elastic deformation $\mathbf{F}^e$ is available, one can immediately obtain the plastic deformation $\mathbf{F}^p$ by

$$\mathbf{F}^p = \mathbf{F}^{e-1}\mathbf{F}. \tag{50}$$

Now we consider the rate quantities related to $\mathbf{F}^e$ and $\mathbf{F}^p$. Let

$$\mathbf{L}^e = \dot{\mathbf{F}}^e\mathbf{F}^{e-1}, \tag{51}$$

$$\mathbf{L}^p = \dot{\mathbf{F}}^p\mathbf{F}^{p-1}. \tag{52}$$

Then, we have

$$\mathbf{L}^e = \mathrm{sym}(\dot{\mathbf{F}}^e\mathbf{F}^{e-1}) + \mathrm{skw}(\dot{\mathbf{F}}^e\mathbf{F}^{e-1}) = \mathbf{D}^e + \mathbf{W}^e, \tag{53}$$

where $\mathbf{D}^e$ is given by Eq. (23) and $\mathbf{W}^e$ by

$$\mathbf{W}^e = \mathrm{skw}(\dot{\mathbf{F}}^e\mathbf{F}^{e-1}) = \mathrm{skw}(\dot{\mathbf{V}}^e\mathbf{V}^{e-1} + \mathbf{V}^e\boldsymbol{\Omega}^e\mathbf{V}^{e-1}), \tag{54}$$

where $\mathbf{V}^e$, $\dot{\mathbf{V}}^e$ and $\boldsymbol{\Omega}^e$ are, respectively, given by Eqs. (46), (49) and (48). Moreover, from Eqs. (39) and (51)–(53), we derive

$$\mathbf{L}^p = \mathbf{F}^{e-1}(\mathbf{L} - \mathbf{L}^e)\mathbf{F}^e = \mathbf{F}^{e-1}(\mathbf{L} - \mathbf{D}^e - \mathbf{W}^e)\mathbf{F}^e. \tag{55}$$

Hence, we have

$$\mathbf{D}^p = \mathrm{sym}\ \mathbf{L}^p = \mathrm{sym}(\mathbf{F}^{e-1}(\mathbf{L} - \mathbf{D}^e - \mathbf{W}^e)\mathbf{F}^e), \tag{56}$$

$$\mathbf{W}^{\mathrm{p}} = \mathrm{skw}\ \mathbf{L}^{\mathrm{p}} = \mathrm{skw}(\mathbf{F}^{\mathrm{e}-1}(\mathbf{L} - \mathbf{D}^{\mathrm{e}} - \mathbf{W}^{\mathrm{e}})\mathbf{F}^{\mathrm{e}}). \tag{57}$$

From the above analysis, we conclude that, within the context of the finite deformation elastoplasticity-damage theory suggested in this and the last sections, the elastic deformation $\mathbf{F}^{\mathrm{e}}$ and the plastic deformation $\mathbf{F}^{\mathrm{p}}$ and all their related kinematical quantities such as the spins $\mathbf{W}^{\mathrm{e}}$ and $\mathbf{W}^{\mathrm{p}}$, etc. can be consistently and uniquely determined. Moreover, it is shown in Xiao et al. (2000) that in a full sense the proposed combination of the two widely used decompositions concerning the total stretching $\mathbf{D}$ and the total deformation gradient $\mathbf{F}$ obeys the invariance requirement under the change of frame or under the superposed rigid body rotation.

5. Rate type and integral type constitutive formulations in the logarithmic rotating frame

In the logarithmic rotating frame specified by Eq. (15) an objective scalar keeps unaltered, whereas an objective symmetric second-order tensor $\mathbf{A}$ becomes $\mathbf{R}^{\log} * \mathbf{A}$. In this section, we denote

$$\mathbf{A}^+ = \mathbf{R}^{\log} * \mathbf{A}. \tag{58}$$

Moreover, we denote

$$\begin{cases} f^+ = \hat{f}(\boldsymbol{\tau}^+, \boldsymbol{\alpha}^+, k, \phi), \qquad g^+ = \hat{g}(\boldsymbol{\tau}^+, \boldsymbol{\alpha}^+, k, \phi), \\ \boldsymbol{\phi}^+ = \hat{\boldsymbol{\phi}}(\boldsymbol{\tau}^+, \boldsymbol{\alpha}^+, k, \phi), \qquad \mathbf{k}^+ = \hat{\mathbf{k}}(\boldsymbol{\tau}^+, \boldsymbol{\alpha}^+, k, \phi), \\ \mathbb{H}^+ = \hat{\mathbb{H}}(\boldsymbol{\tau}^+, \boldsymbol{\alpha}^+, k, \phi). \end{cases} \tag{59a, b, c}$$

The Eulerian rate type constitutive formulation proposed in Section 3 is frame-indifferent, and hence, its form in the logarithmic rotating frame specified by Eq. (15) remains the same. Consequently, in the logarithmic frame Eqs. (28)–(30) and (37) become

$$\dot{\phi} = \dot{\lambda}\boldsymbol{\phi}^+ : \frac{\partial g^+}{\partial \boldsymbol{\tau}^+}, \tag{60}$$

$$\dot{k} = \dot{\lambda}\mathbf{k}^+ : \frac{\partial g^+}{\partial \boldsymbol{\tau}^+}, \tag{61}$$

$$\overset{\circ}{\boldsymbol{\alpha}}{}^{\log +} = \dot{\lambda}\mathbb{H}^+ : \frac{\partial g^+}{\partial \boldsymbol{\tau}^+},$$

$$\mathbf{D}^+ = \frac{\partial^2 \Sigma^+}{\partial \bar{\boldsymbol{\tau}}^+ \partial \bar{\boldsymbol{\tau}}^+} : \overset{\circ}{\bar{\boldsymbol{\tau}}}{}^{\log +} + \dot{\lambda}\frac{\partial g^+}{\partial \boldsymbol{\tau}^+}.$$

Applying Eqs. (58), (13) and (14), we rewrite the last two equations into the forms

$$\dot{\overline{\boldsymbol{\alpha}^+}} = \dot{\lambda}\mathbb{H}^+ : \frac{\partial g^+}{\partial \boldsymbol{\tau}^+}, \tag{62}$$

$$\dot{\overline{\mathbf{h}^+}} = \dot{\overline{\left(\frac{\partial \Sigma^+}{\partial \bar{\boldsymbol{\tau}}^+}\right)}} + \dot{\lambda}\frac{\partial g^+}{\partial \boldsymbol{\tau}^+}. \tag{63}$$

Moreover, the plastic multiplier $\dot{\lambda}$ given by Eq. (35) becomes

$$\begin{cases} \dot{\lambda} = -\dfrac{\psi}{\beta}\dfrac{\partial f^+}{\partial \boldsymbol{\tau}^+} : \dot{\overline{\boldsymbol{\tau}^+}}, \\ \beta = \dfrac{\partial f^+}{\partial \boldsymbol{\alpha}^+} : \mathbb{H}^+ : \dfrac{\partial g^+}{\partial \boldsymbol{\tau}^+} + \dfrac{\partial f^+}{\partial k}\left(\mathbf{k}^+ : \dfrac{\partial g^+}{\partial \boldsymbol{\tau}^+}\right) + \dfrac{\partial f^+}{\partial \phi}\left(\boldsymbol{\phi}^+ : \dfrac{\partial g^+}{\partial \boldsymbol{\tau}^+}\right), \end{cases} \tag{64}$$

where

$$\psi = \begin{cases} 0 & \text{if } f^+ < 0 \text{ or } f^+ = 0 \text{ and } \dfrac{\partial \hat{f}^+}{\partial \boldsymbol{\tau}^+} : \dot{\overline{\boldsymbol{\tau}^+}} < 0, \\ 1 & \text{if } f^+ = 0 \text{ and } \dfrac{\partial \hat{f}^+}{\partial \boldsymbol{\tau}^+} : \dot{\overline{\boldsymbol{\tau}^+}} \geqslant 0. \end{cases} \tag{65}$$

Eqs. (60)–(65) supply the forms of the rate constitutive equations (28)–(30) and (35)–(37) in the logarithmic rotating frame. It turns out that the rate equations (60)–(65) have the same structure as the counterpart of small deformation elastoplastic damage theory. Indeed, whenever the deformation is small, the logarithmic strain measure $\mathbf{h}$ and the logarithmic rotation $\mathbf{R}^{\log}$ approximate to the small strain measure $\boldsymbol{\epsilon}$ and the identity tensor $\mathbf{I}$, respectively, and accordingly the quantities $\boldsymbol{\tau}^+$ and $\boldsymbol{\alpha}^+$ and $\mathbf{h}^+$ to $\boldsymbol{\tau}$ and $\boldsymbol{\alpha}$ and $\boldsymbol{\epsilon}$, respectively. With these approximations Eqs. (60)–(65) are reduced to the rate constitutive equations for small deformation elastoplastic damage.

Owing to the fact indicated above, the numerical integration of Eqs. (60)–(65) formulated in the logarithmic rotating frame may be carried out by means of the numerical methods developed for small deformation theory. Then, the quantities in the current configuration are available from the corresponding quantities in the logarithmic rotating frame and the logarithmic rotation $\mathbf{R}^{\log}$, the latter being obtained by integrating Eq. (11) with the initial condition (12).

In addition, in the logarithmic rotating frame, Eqs. (14) and (45) are of the forms

$$\dot{\overline{\mathbf{h}^+}} = \mathbf{D}^+, \tag{66}$$

$$\dot{\overline{\mathbf{h}^{e+}}} = \mathbf{D}^{e+}. \tag{67}$$

The former, being a rigorous kinematical relation, simply means that in the logarithmic rotating frame the total stretching is exactly the material time derivative of the total logarithmic strain measure, whereas the latter, which defines the elastic part $\mathbf{D}^e$ of the total stretching $\mathbf{D}$, implies that in the logarithmic rotating frame, the material time derivative of the elastic logarithmic strain measure supplies the elastic part of the total stretching. These show that the relationship (45) is a natural and consistent definition motivated by and based on the rigorous kinematical relation (14).

Finally, integrating Eqs. (60)–(65) over the time interval $[0, t]$ and using the equalities

$$\frac{\partial h^+}{\partial \boldsymbol{\tau}^+} = \mathbf{R}^{\log} * \frac{\partial h}{\partial \boldsymbol{\tau}}, \qquad h = f, g, \Sigma,$$

$$\mathbf{A}^+ = \mathbf{R}^{\log} * \mathbf{A}, \qquad \mathbf{A} = \boldsymbol{\phi}, \mathbf{k}, \mathbb{H},$$

as well as the last two identities in Section 1, the initial condition (16) and $\boldsymbol{\alpha}\,|_{t=0} = \mathbf{O}$, we arrive at

$$\phi = \phi|_{t=0} + \int_0^t \dot{\lambda} \boldsymbol{\phi} : \frac{\partial g}{\partial \boldsymbol{\tau}}\ \mathrm{d}s, \tag{68}$$

$$k = \int_0^t \dot{\lambda} \mathbf{k} : \frac{\partial g}{\partial \boldsymbol{\tau}} \mathrm{d}s, \tag{69}$$

$$\boldsymbol{\alpha} = \mathbf{R}^{\log \mathrm{T}} * \left(\int_0^t \mathbf{R}^{\log} * \left(\dot{\lambda} \mathbb{H} : \frac{\partial g}{\partial \boldsymbol{\tau}} \right) \mathrm{d}s \right), \tag{70}$$

$$\mathbf{h} = \frac{\partial \Sigma}{\partial \bar{\boldsymbol{\tau}}} + \mathbf{R}^{\log \mathrm{T}} * \left(\int_0^t \dot{\lambda} \mathbf{R}^{\log} * \frac{\partial g}{\partial \boldsymbol{\tau}} \mathrm{d}s \right). \tag{71}$$

Through the logarithmic rotation $\mathbf{R}^{\log}$, the above integral type formulation indicates, in a clear and direct manner, the effect of the finite strain and rotation history on the current stress, the damage and the hardening behaviour.

6. A model for void growth and nucleation in metals

In this section, we apply the general model established in the previous sections to derive a model for void growth and nucleation in porous metals at finite deformations. In this case, the damage variable ϕ is interpreted as the void volume fraction.

Based on certain simplified assumptions, Gurson (1977) was the first to establish a continuum model for void growth and nucleation in porous ductile media with perfectly plastic matrix. Gurson's model was modified and developed later by various researchers, refer to, e.g. Tvergaard (1982a,b), Tvergaard and Needleman (1984), and Meer and Hutchinson (1985). This aspect is mentioned in the review articles by Neale (1981) and Nemat-Nasser (1992) and discussed by Voyiadjis and Kattan (1992a,b) (some relevant remarks on the latter will be made at the end of this section). Taking these subsequent modifications into account, we here assume a modified form of Gurson's yield function as follows (cf. Eq. (89) in Voyiadjis and Kattan 1992a):

$$f = \frac{3}{2}(\boldsymbol{\tau}' - \boldsymbol{\alpha}) : (\boldsymbol{\tau}' - \boldsymbol{\alpha}) + 2q_1\sigma_{\mathrm{F}}^2\phi\,\mathrm{ch}\left(\frac{\mathrm{tr}\,\boldsymbol{\tau}}{2\sigma_{\mathrm{F}}}\right) - \sigma_{\mathrm{F}}^2(1 + q_2\phi^2), \tag{72}$$

where $\boldsymbol{\tau}'$ is the deviatoric Kirchhoff stress, σ_{F} is the flow yield strength of the metal matrix at uniaxial tensile test, and q_1 and q_2 are the two modified material parameters introduced by Tvergaard (1982a,b) and Tvergaard and Needleman (1984). Moreover, the back stress, which defines the centre of the current yield surface, is assumed to be traceless, i.e.

$$\mathrm{tr}\,\boldsymbol{\alpha} = 0. \tag{73}$$

Throughout, $\mathrm{ch}\,x$ and $\mathrm{sh}\,x$ are used to denote the hyperbolic cosine and sine functions of x.

As commonly done, we assume an associated flow rule and the kinematic hardening rule of Prager–Ziegler's type. Thus, we have $f \equiv g$ and

$$\overset{\circ}{\boldsymbol{\alpha}}{}^{\log} = \dot{\lambda}c(\boldsymbol{\tau}' - \boldsymbol{\alpha}), \tag{74}$$

with c being a kinematic hardening parameter. It is easy to demonstrate that this is just a particular form of the general evolution equation (30). Besides, the flow rule (26) becomes

$$\mathbf{D}^{\mathrm{ep}} = \dot{\lambda}\frac{\partial f}{\partial \boldsymbol{\tau}}. \tag{75}$$

Usually, the elastic strain in a metal matrix is small. In this case, we can take the gradient $\partial^2\Sigma/\partial\bar{\boldsymbol{\tau}}\,\partial\bar{\boldsymbol{\tau}}$ as the constant isotropic compliance tensor given by Eq. (21). Hence, Eqs. (23) and (44) for the damaged elastic response become

$$\mathbf{D}^{\mathrm{e}} = \mathbb{D} : \overset{\circ}{\boldsymbol{\tau}}{}^{\log}, \tag{76}$$

$$\mathbf{h}^{\mathrm{e}} = \mathbb{D} : \bar{\boldsymbol{\tau}} = \frac{\mathbb{D}}{1 - \phi} : \boldsymbol{\tau}, \tag{77}$$

where $\mathbb{D}$ is given by Eq. (21).

During the course of deformation, both the growth of existing voids and the nucleation of new cavities contribute to the change of the void volume fraction ϕ. Following Needleman and Rice (1978), we assume the evolution equation of the void volume fraction ϕ as follows:

$$\dot{\phi} = (1-\phi)\mathrm{tr}\,\mathbf{D}^{\mathrm{ep}} + A\dot{\sigma}_{\mathrm{F}}. \tag{78}$$

In the above, the first term of the right-hand side arises from the contribution of the void growth and the second term from the contribution of the void nucleation, with A being a material parameter. Here, we assume that the void nucleation is correlated directly with the flow yield strength σ_{F}.

On the other hand, from the equivalence of the overall rate of plastic work and that in the matrix material the following relation is derived (see Eqs. (2.38b) and (2.36c) given in Nemat-Nasser (1992)):

$$\dot{\sigma}_{\mathrm{F}} = b\dot{\sigma}_{\mathrm{Y}} = \delta\frac{\boldsymbol{\tau}:\mathbf{D}^{\mathrm{ep}}}{(1-\phi)\sigma_{\mathrm{F}}} = \dot{\lambda}\frac{\delta}{(1-\phi)\sigma_{\mathrm{F}}}\left(\boldsymbol{\tau}:\frac{\partial f}{\partial\boldsymbol{\tau}}\right), \tag{79}$$

with

$$\delta = b\frac{EE_{\mathrm{t}}}{E-E_{\mathrm{t}}}, \qquad \sigma_{\mathrm{F}} = (1-b)\sigma_{\mathrm{Y}}^{0} + b\sigma_{\mathrm{Y}}, \tag{80}$$

where σ_{Y}^{0} is the initial and σ_{Y} the current flow stress for the metal matrix, and the parameter b ranges from 0 to 1 with $b = 1, 0$ corresponding to, separately, purely isotropic and purely kinematic hardening. Besides, E and E_{t} are Young's modulus and the tangent modulus associated with the Kirchhoff stress-logarithmic strain curve in a uniaxial test of the metal matrix. Here σ_{Y} characterizes the isotropic hardening of metal matrix. Hence, by identifying the internal variable k with σ_{F}, Eq. (79) is again a particular form of the general evolution equation (29).

Combining Eqs. (78) and (79), and using Eq. (75), we derive the evolution equation of the void volume fraction ϕ as follows:

$$\dot{\phi} = \dot{\lambda}\left((1-\phi)\,\mathrm{tr}\left(\frac{\partial f}{\partial\boldsymbol{\tau}}\right) + \frac{A\delta}{(1-\phi)\sigma_{\mathrm{F}}}\left(\boldsymbol{\tau}:\frac{\partial f}{\partial\boldsymbol{\tau}}\right)\right). \tag{81}$$

Furthermore, Eqs. (75) and (76) produce

$$\mathbf{D} = \mathbb{D} : \overset{\circ}{\boldsymbol{\tau}}{}^{\log} + \dot{\lambda}\frac{\partial f}{\partial\boldsymbol{\tau}}. \tag{82}$$

Eqs. (82), (74), (79) and (81), together with Eqs. (A.5)–(A.7), supply a system of Eulerian rate type constitutive equations governing the Kirchhoff stress $\boldsymbol{\tau}$, the back stress $\boldsymbol{\alpha}$, the flow strength σ_{F} and the void volume fraction ϕ. This system and the consistent combination of the decompositions (18) and (38), proposed in Section 4, constitute an Eulerian rate type model for void growth and nucleation in porous metals experiencing finite elastic–plastic deformations. As a test problem, in Section 7 we study the finite simple shear response of this model by means of Runge–Kutta numerical integration.

We conclude this section with some remarks on recent interesting and instructive works by Voyiadjis and Kattan (1992a,b). In these works, a general large deformation elasto-plasticity-damage theory with a symmetric second-order damage tensor variable has been established by postulating the decomposition (18) and adopting the corotational rates defined by the spin tensors of the form

$$\boldsymbol{\Omega} = \omega\mathbf{W},$$

with ω being a scalar influence parameter. The general theory is applied to derive a model for void growth by relating a quadratic yield function of von Mises type with combined isotropic-kinematic hardening to a modified Gurson's yield function. Some interesting results have been obtained in this case. The idea and approach employed are insightful, instructive and quite general. However, here we would like to raise two questions for consideration.

First, a corotational rate $\overset{\circ}{\mathbf{S}}$ defined by a spin tensor $\boldsymbol{\Omega}$ of the aforementioned form, i.e.

$$\overset{\circ}{\mathbf{S}} = \dot{\mathbf{S}} + \mathbf{S}\boldsymbol{\Omega} - \boldsymbol{\Omega}\mathbf{S}, \qquad \boldsymbol{\Omega} = \omega\mathbf{W},$$

is *not objective except for the case when* $\omega = 1$. To substantiate this statement, consider the transformation of the above corotational rate under the change of frame specified by a time-dependent proper orthogonal tensor $\mathbf{Q}$. Under the just-mentioned change of frame, an objective symmetric second-order Eulerian tensor $\mathbf{S}$ changes to $\mathbf{QSQ}^{\mathrm{T}}$, while the vorticity tensor $\mathbf{W}$ changes to $\mathbf{QWQ}^{\mathrm{T}} + \dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}}$ (see, e.g. Ogden (1984)) and hence the spin $\boldsymbol{\Omega}$ to

$$\mathbf{Q}\boldsymbol{\Omega}\mathbf{Q}^{\mathrm{T}} + \omega\dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}}.$$

In addition, an objective scalar ω, in particular, a constant ω, keeps unaltered. As a result, the corotational rate $\overset{\circ}{\mathbf{S}}$ changes to

$$\overline{\left(\mathbf{QSQ}^{\mathrm{T}}\right)}^{\boldsymbol{\cdot}} + \left(\mathbf{QSQ}^{\mathrm{T}}\right)\left(\mathbf{Q}\boldsymbol{\Omega}\mathbf{Q}^{\mathrm{T}} + \omega\dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}}\right) - \left(\mathbf{Q}\boldsymbol{\Omega}\mathbf{Q}^{\mathrm{T}} + \omega\dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}}\right)\left(\mathbf{QSQ}^{\mathrm{T}}\right).$$

Then, by the virtue of Eq. (3) and the equality

$$\dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}} = -\mathbf{Q}\dot{\mathbf{Q}}^{\mathrm{T}},$$

one can further deduce that the corotational rate $\overset{\circ}{\mathbf{S}}$ changes to

$$\mathbf{Q}\overset{\circ}{\mathbf{S}}\mathbf{Q}^{\mathrm{T}} + (1-\omega)(\dot{\mathbf{Q}}\mathbf{S}\mathbf{Q}^{\mathrm{T}} + \mathbf{QS}\dot{\mathbf{Q}}^{\mathrm{T}}).$$

Thus, the corotational rate $\overset{\circ}{\mathbf{S}}$ is objective, i.e. the latter is identical to $\mathbf{Q}\overset{\circ}{\mathbf{S}}\mathbf{Q}^{\mathrm{T}}$ for every time-dependent proper orthogonal tensor $\mathbf{Q}$, if and only if

$$(1-\omega)(\dot{\mathbf{Q}}\mathbf{S}\mathbf{Q}^{\mathrm{T}} + \mathbf{QS}\dot{\mathbf{Q}}^{\mathrm{T}}) = \mathbf{O}$$

for every time-dependent proper orthogonal tensor $\mathbf{Q}$. The latter is possible only for the case when $\omega = 1$, i.e., $\overset{\circ}{\mathbf{S}}$ is the well-known Zaremba–Jaumann rate.

Next, through relating the general model to a modified Gurson's model, the material parameters q_1 and q_2 in the modified Gurson's yield function given by Eq. (72) are found to assume the forms (cf. Eqs. (100) and (83) in Voyiadjis and Kattan (1992a,b), respectively; the void volume fraction v therein has been replaced by ϕ here):

$$q_1 = \frac{4}{3\phi}, \quad q_2 = \frac{8}{3\phi^2}.$$

Substituting the above expressions into Eq. (72), one arrives at

$$f = \frac{2}{3}(\boldsymbol{\tau}' - \boldsymbol{\alpha}) : (\boldsymbol{\tau}' - \boldsymbol{\alpha}) + \frac{8}{3}\sigma_{\mathrm{F}}^2 \operatorname{ch}\left(\frac{\operatorname{tr}\boldsymbol{\tau}}{2\sigma_{\mathrm{F}}}\right) - \frac{11}{3}\sigma_{\mathrm{F}}^2.$$

The latter, however, implies that the void volume fraction ϕ has no influence on yielding behaviour.

7. Finite simple shear

For the sake of simplicity, we consider Gurson's model with purely isotropic hardening, i.e.

$$q_1 = q_2 = 1, \quad b = 1, \quad \sigma_{\mathrm{F}} = \sigma_{\mathrm{Y}}, \quad c = 0, \quad \boldsymbol{\alpha} = \mathbf{O}.$$

Moreover, it will be shown shortly that in the course of finite simple shear deformation, the spherical component of the stress $\mathrm{tr}\,\boldsymbol{\tau}$ vanishes, i.e.

$$\mathrm{tr}\,\boldsymbol{\tau} = 0, \quad \boldsymbol{\tau}' = \boldsymbol{\tau}.$$

Hence, for the case at issue, the yield condition is of the form

$$\bar{\boldsymbol{\tau}} : \bar{\boldsymbol{\tau}} - \tfrac{2}{3}\sigma_{\mathrm{Y}}^2 = 0, \tag{83}$$

and Eqs. (A.5)–(A.8) and (A.11) are reduced to

$$\dot{\phi} = A\dot{\sigma}_{\mathrm{Y}}, \tag{84}$$

$$\dot{\sigma}_{\mathrm{Y}} = 2\dot{\lambda}\delta(1-\phi)\sigma_{\mathrm{Y}}, \tag{85}$$

$$\mathbf{D} = \frac{1}{2G}\overset{\circ}{\bar{\boldsymbol{\tau}}}{}^{\log} + 3\dot{\lambda}\boldsymbol{\tau}, \tag{86}$$

where the plastic multiplier $\dot{\lambda}$ is given by

$$\dot{\lambda} = \frac{\psi}{2\delta}\,\frac{1}{(1-\phi-A\sigma_Y)}\,\frac{\boldsymbol{\tau} : \dot{\boldsymbol{\tau}}}{\boldsymbol{\tau} : \boldsymbol{\tau}}. \tag{87}$$

For simple shear deformation, we have $\mathrm{tr}\,\mathbf{D} = 0$. From this and Eq. (86), we infer $\mathrm{tr}\,\dot{\boldsymbol{\tau}} = 0$. Then, the latter and Eq. (16b) gives $\mathrm{tr}\,\boldsymbol{\tau} = 0$, as mentioned before.

Taking into account now the relation (cf. Eq. (17))

$$\dot{\boldsymbol{\tau}} = (1-\phi)\dot{\bar{\boldsymbol{\tau}}} - \dot{\phi}\bar{\boldsymbol{\tau}},$$

the plastic multiplier $\dot{\lambda}$ can be redefined for the effective Kirchhoff stress $\bar{\boldsymbol{\tau}}$

$$\dot{\lambda} = \frac{\psi}{2\delta}\,\frac{1}{(1-\phi)}\,\frac{\bar{\boldsymbol{\tau}} : \dot{\bar{\boldsymbol{\tau}}}}{\bar{\boldsymbol{\tau}} : \bar{\boldsymbol{\tau}}}. \tag{88}$$

For processes with prescribed deformation, as it is the case for simple shear, it is more convenient to express here the rate of the effective stress through the stretching. This can be achieved by multiplying Eq. (86) with $\bar{\boldsymbol{\tau}}$

$$\bar{\boldsymbol{\tau}} : \mathbf{D} = \frac{1}{2G}\,\bar{\boldsymbol{\tau}} : \dot{\bar{\boldsymbol{\tau}}} + 3\dot{\lambda}\bar{\boldsymbol{\tau}} : \boldsymbol{\tau}.$$

We introduce here Eq. (88), and finally find

$$\bar{\boldsymbol{\tau}} : \dot{\bar{\boldsymbol{\tau}}} = \frac{2G\delta}{\delta + \psi 3G}\,\bar{\boldsymbol{\tau}} : \mathbf{D}. \tag{89}$$

With this transformation, the rate equations (85) and (86) can be redefined to give

$$\dot{\sigma}_{\mathrm{Y}} = \psi\,\frac{3G\delta}{\delta + 3G}\,\frac{\bar{\boldsymbol{\tau}} : \mathbf{D}}{\sigma_{\mathrm{Y}}}, \tag{90}$$

$$\overset{\circ}{\bar{\boldsymbol{\tau}}}{}^{\log} = \dot{\bar{\boldsymbol{\tau}}} + \bar{\boldsymbol{\tau}}\boldsymbol{\Omega}^{\log} - \boldsymbol{\Omega}^{\log}\bar{\boldsymbol{\tau}} = 2G\left(\mathbf{D} - \frac{3}{2}\psi\,\frac{3G}{\delta + 3G}\,\frac{\bar{\boldsymbol{\tau}} : \mathbf{D}}{\sigma_{\mathrm{Y}}^2}\bar{\boldsymbol{\tau}}\right). \tag{91}$$

The finite simple shear deformation is specified by

$$\mathbf{x} = (X_1 + \gamma X_2)\mathbf{e}_1 + X_2\mathbf{e}_2 + X_3\mathbf{e}_3, \tag{92}$$

where $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is a fixed orthonormal basis; $\mathbf{X} = \sum_{i=1}^{3} X_i \mathbf{e}_i$ and $\mathbf{x} = \sum_{i=1}^{3} x_i \mathbf{e}_i$ are the initial and the current position vectors of a material particle, respectively.

The left Cauchy–Green tensor $\mathbf{B}$ is of the form

$$\mathbf{B} = (1+\gamma^2)\mathbf{e}_1 \otimes \mathbf{e}_1 + \gamma(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1) + \mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3. \tag{93}$$

The eigenvalues of $\mathbf{B}$ are as follows:

$$\begin{cases} \chi_1 = (2+\gamma^2+\gamma\sqrt{4+\gamma^2})/2, \\ \chi_2 = (2+\gamma^2-\gamma\sqrt{4+\gamma^2})/2 = (\chi_1)^{-1}, \\ \chi_3 = 1. \end{cases} \tag{94}$$

The logarithmic strain $\mathbf{h}$ for simple shear is of the form

$$\mathbf{h} = \frac{1}{2}\ln \mathbf{B} = \frac{\mathrm{sh}^{-1}\omega}{\sqrt{1+\omega^2}}(\omega(\mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{e}_2 \otimes \mathbf{e}_2) + \mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1). \tag{95}$$

Here and hereafter $\omega = \gamma/2$, and $\mathrm{sh}^{-1}\,\omega$ is used to represent the inverse hyperbolic sine function of ω, i.e.

$$\mathrm{sh}^{-1}\,\omega = \ln(\omega + \sqrt{1+\omega^2}).$$

The stretching $\mathbf{D}$ and the vorticity tensor $\mathbf{W}$ are given by

$$\mathbf{D} = \dot{\omega}(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1), \tag{96}$$

$$\mathbf{W} = \dot{\omega}(\mathbf{e}_1 \otimes \mathbf{e}_2 - \mathbf{e}_2 \otimes \mathbf{e}_1). \tag{97}$$

The process of simple shear deformation response consists of two stages: First, the elastic response (hence $\mathbf{D}^{\mathrm{ep}} = \mathbf{O}$) starts at $\omega = 0$ and ends at a yielding point $\omega = \omega^{\mathrm{p}}$, and then follows the elastic–plastic response (hence $\mathbf{D}^{\mathrm{ep}} \neq \mathbf{O}$) for all $\omega \geqslant \omega^{\mathrm{p}}$. The two stages will be studied separately.

7.1. The elastic response and the elastic–plastic transition

For the elastic response, we have $\mathbf{h}^{\mathrm{e}} = \mathbf{h}$. Hence, Eq. (77) yields

$$\frac{\boldsymbol{\tau}}{2G} = \frac{1}{2}\ln \mathbf{B} = \frac{\mathrm{sh}^{-1}\,\omega}{\sqrt{1+\omega^2}}(\omega(\mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{e}_2 \otimes \mathbf{e}_2) + \mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1). \tag{98}$$

In deriving the above, $\ln(\det \mathbf{B}) = \ln(\chi_1\chi_2\chi_3) = 0$ is used. Moreover, due to $\psi = 0$ (cf. Eq. (36)), there is no void nucleation and growth, i.e.

$$\phi = 0$$

during the whole stage of elastic response, if $\phi|_{t=0} = 0$. Thus, we have

$$\frac{\tau_{12}}{2G} = \frac{\mathrm{sh}^{-1}\,\omega}{\sqrt{1+\omega^2}}, \tag{99}$$

$$\frac{\tau_{11}}{2G} = -\frac{\tau_{22}}{2G} = \omega\frac{\mathrm{sh}^{-1}\,\omega}{\sqrt{1+\omega^2}}, \qquad \tau_{11} = \omega\tau_{12}. \tag{100}$$

The above elastic response starts at $\omega = 0$ and concludes with

$$\boldsymbol{\tau} : \boldsymbol{\tau} = \tfrac{2}{3}(\sigma_{\mathrm{Y}}^{0})^2,$$

i.e.

$$\tau_{12}^2 + \tau_{11}^2 = \tfrac{1}{3}(\sigma_Y^0)^2, \tag{101}$$

which corresponds to the yield point $\omega = \omega^p$. Using the expressions (99)–(101), we infer that ω^p is determined by

$$\mathrm{sh}^{-1}\,\omega^p = \frac{\sqrt{3}\sigma_Y^0}{6G}.$$

Hence, we have

$$\omega^p = \mathrm{sh}\frac{\sqrt{3}\sigma_Y^0}{6G}. \tag{102}$$

Let further $\tau_{ij}^p = \tau_{ij}|_{\omega=\omega^p}$, then we have

$$\frac{\tau_{12}^p}{2G} = \frac{\sqrt{3}\sigma_Y^0/6G}{\sqrt{1+\mathrm{sh}^2(\sqrt{3}\sigma_Y^0/6G)}}, \tag{103}$$

$$\frac{\tau_{11}^p}{2G} = -\frac{\tau_{22}^p}{2G} = \frac{(\sqrt{3}\sigma_Y^0/6G)\,\mathrm{sh}(\sqrt{3}\sigma_Y^0/6G)}{\sqrt{1+\mathrm{sh}^2(\sqrt{3}\sigma_Y^0/6G)}}. \tag{104}$$

In particular, following Moss (1984) (cf. Eq. (29) therein, in which $Y_0/G = 0.1$ with Y_0 being σ_Y^0 here), we choose the initial yield stress σ_Y^0 as

$$\sigma_Y^0/G = 0.1. \tag{105}$$

For such a case, Eq. (102) yields

$$\omega^p = \mathrm{sh}\frac{\sqrt{3}}{60} = 2.8871\times 10^{-2},$$

and hence

$$\frac{\tau_{12}^p}{2G} = 2.8855\times 10^{-2}, \qquad \frac{\tau_{11}^p}{2G} = -\frac{\tau_{22}^p}{2G} = 8.3310\times 10^{-4}.$$

7.2. The elastoplastic response and the void nucleation

The plastic flow and the void nucleation occurs whenever $\omega \geqslant \omega^p$ and are governed by the yield condition (83) and the rate equations (84), (90) and (91), with the loading–unloading indicator $\psi = 1$.

Note that $\bar{\boldsymbol{\tau}}$ is the effective Kirchhoff stress. For simple shear deformation, however, the latter is identical with the effective Cauchy stress $\boldsymbol{\sigma}/(1-\phi)$, due to the fact (cf. Eq. (93))

$$\det \mathbf{F} = \sqrt{\det \mathbf{B}} = 1.$$

The above system may be further simplified. In fact, for the simple shear deformation (cf. Eq. (92)), both the left Cauchy–Green tensor $\mathbf{B}$ (cf. Eq. (93)) and the stretching $\mathbf{D}$ (cf. Eq. (96)) are essentially symmetric second-order tensors in two dimensions. As a result, the expression $(2.9)_3$ in Xiao et al. (1997a) can be reduced to Eq. $(2.9)_2$ therein, i.e. the log-spin $\boldsymbol{\Omega}^{\log}$ for the simple shear deformation is given by

$$\boldsymbol{\Omega}^{\log} = \mathbf{W} + v(\mathbf{BD} - \mathbf{DB}) = \dot{\omega}(1+4v\omega^2)(\mathbf{e}_1\otimes\mathbf{e}_2 - \mathbf{e}_2\otimes\mathbf{e}_1),$$

where the coefficient v can be obtained by using Eq. (2.10) in Xiao et al. (1997a) and Eq. (94). Hence, we have

$$\boldsymbol{\Omega}^{\log} = \frac{1}{2}\dot{\omega}\left(\frac{1}{1+\omega^2} + \frac{\omega}{\sqrt{1+\omega^2}\,\mathrm{sh}^{-1}\,\omega}\right)(\mathbf{e}_1 \otimes \mathbf{e}_2 - \mathbf{e}_2 \otimes \mathbf{e}_1). \tag{106}$$

Then, by means of Eqs. (91), (96) and (106), we infer

$$\dot{x} + \frac{6G^2}{\delta + 3G}\frac{\bar{\boldsymbol{\tau}} : \mathbf{D}}{\bar{\boldsymbol{\tau}} : \bar{\boldsymbol{\tau}}}x = 0$$

with $x = \bar{\tau}_{33},\ \bar{\tau}_{13},\ \bar{\tau}_{23},\ \bar{\tau}_{11} + \bar{\tau}_{22}$. Since each x, vanishes during the whole elastic process, from the above differential equation for x, we infer that each x continues to vanish during the whole stage of the succeeding elastic–plastic process. Thus, the effective stress $\bar{\boldsymbol{\tau}}$ is of the form

$$\bar{\boldsymbol{\tau}} = \bar{\tau}_{11}(\mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{e}_2 \otimes \mathbf{e}_2) + \bar{\tau}_{12}(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1). \tag{107}$$

Utilizing the latter and the identity

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \dot{\omega}\frac{\mathrm{d}\varphi}{\mathrm{d}\omega}$$

for each function $\varphi = \varphi(\omega)$, we arrive at simplified forms of Eqs. (83) and (91):

$$\sigma'_{\mathrm{Y}} = \sqrt{3}\sqrt{\bar{\tau}'^2_{11} + \bar{\tau}'^2_{12}}, \tag{108}$$

$$\frac{\mathrm{d}\bar{\tau}'_{11}}{\mathrm{d}\omega} - \left(\frac{1}{1+\omega^2} + \frac{\omega}{\sqrt{1+\omega^2}\ \mathrm{sh}^{-1}\,\omega}\right)\bar{\tau}'_{12} + \frac{\varrho\bar{\tau}'_{11}\bar{\tau}'_{12}}{\bar{\tau}'^2_{11} + \bar{\tau}'^2_{12}} = 0, \tag{109}$$

$$\frac{\mathrm{d}\bar{\tau}'_{12}}{\mathrm{d}\omega} + \left(\frac{1}{1+\omega^2} + \frac{\omega}{\sqrt{1+\omega^2}\ \mathrm{sh}^{-1}\,\omega}\right)\bar{\tau}'_{11} - \frac{\bar{\tau}'^2_{11} + (1-\varrho)\bar{\tau}'^2_{12}}{\bar{\tau}'^2_{11} + \bar{\tau}'^2_{12}} = 0. \tag{110}$$

Here and henceforth,

$$\varrho = \frac{3G}{\delta + 3G}, \quad \sigma'_{\mathrm{Y}} = \frac{\sigma_{\mathrm{Y}}}{2G}, \quad \bar{\tau}'_{ij} = \frac{\bar{\tau}_{ij}}{2G}, \quad \tau'_{ij} = \frac{\tau_{ij}}{2G}.$$

Eqs. (109) and (110) with the initial conditions (103) and (104) determine the effective stress components $\bar{\tau}'_{11}$ and $\bar{\tau}'_{12}$ as functions of the shear strain ω. Then, Eq. (108) gives the flow yield strength σ'_{Y}. In addition, the void volume fraction ϕ is obtained by integrating the rate equation (84). Finally, the true stress components τ'_{11} and τ'_{12} are given by

$$\tau'_{11} = -\tau'_{22} = (1-\phi)\bar{\tau}'_{11}, \quad \tau'_{12} = (1-\phi)\bar{\tau}'_{12}. \tag{111}$$

Various forms of the evolution relation of the void fraction volume ϕ to the flow yield strength σ_{Y} are possible, refer to, e.g. Lemaitre and Chaboche (1990). If the material parameter A in Eq. (84) is regarded as a constant, then one can readily derive a linear relation between ϕ and σ_{Y}. Here, we assume the exponential form

$$\phi = 1 - \mathrm{e}^{-k(\sigma_{\mathrm{Y}} - \sigma^0_{\mathrm{Y}})/\sigma^0_{\mathrm{Y}}}, \quad \sigma_{\mathrm{Y}} \geqslant \sigma^0_{\mathrm{Y}}, \tag{112}$$

where $k > 0$ is a dimensionless material parameter. Evidently, whenever the flow yield strength σ_{Y} is close to the initial yield strength σ^0_{Y}, i.e. $(\sigma_{\mathrm{Y}} - \sigma^0_{\mathrm{Y}})/\sigma^0_{\mathrm{Y}}$ is small, the foregoing linear relation gives a good approximation of the general relation (112).

Setting $\varrho = 0.9$ and $\sigma^0_{\mathrm{Y}}/G = 0.1$, we obtain the effective normal stress $\bar{\tau}'_{11}$ and the effective shear stress $\bar{\tau}'_{12}$ by means of Dormand–Prince numerical integration. Then, from Eq. (112), we obtain the void volume fraction ϕ for various possible values of the material parameter k. Here we set $k = 0.2$. Finally, we obtain the true normal stress τ'_{11} and the true shear stress τ'_{12} from Eq. (111).

The results are shown in Figs. 1–5, where in Fig. 1, the dimensionless effective shear stress is depicted vs the shear strain ω. In addition to the solution of the differential equations (109) and (110) for the logarithmic rate, we also have presented solutions for the Zaremba–Jaumann rate and the Green–Naghdi rate, where according to Eq. (5), we have introduced $\boldsymbol{\Omega}^{\mathrm{J}} = \mathbf{W}$ and $\boldsymbol{\Omega}^{\mathrm{GN}} = \dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}}$, respectively. It is illustrated that both results for the logarithmic rate and the Green–Naghdi rate as well show almost linear increasing behaviour, whereas the Zaremba–Jaumann rate tends to display oscillating properties and, for the hardening parameter ϱ under consideration, even changes the sign of the shear stress for very high strains of about $\omega = 8.3$. As has been observed by various authors the Jaumann rate may cause physically not plausible results.

The effective normal stresses vs the shear strain ω are shown in Fig. 2. Here again the Zaremba–Jaumann rate tends to display oscillating behaviour. The Green–Naghdi rate leads to an almost constant normal stress. Only the logarithmic rate creates a monotonically increasing stress as would be expected from experimental observations.

The respective dimensionless true stresses are presented in Figs. 3 and 4. Whereas the shear stresses for the logarithmic rate and the Green–Naghdi rate are almost identical, first increasing and then decreasing,

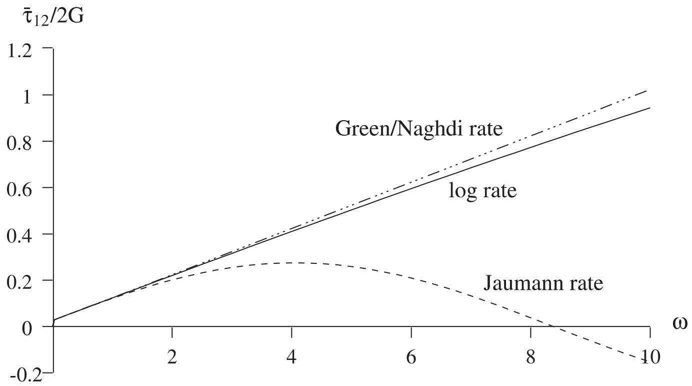

Fig. 1. Effective shear stresses vs shear strain.

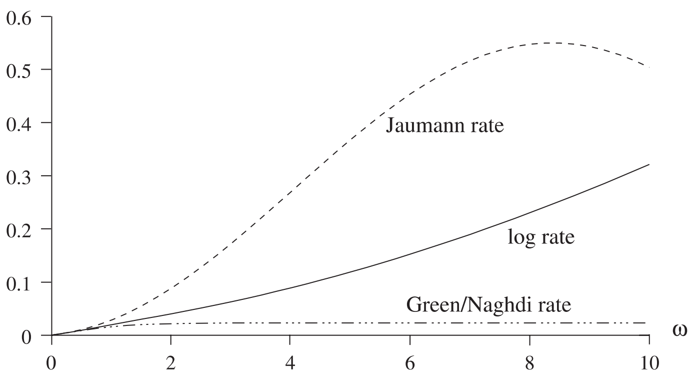

Fig. 2. Effective normal stresses vs shear strain.

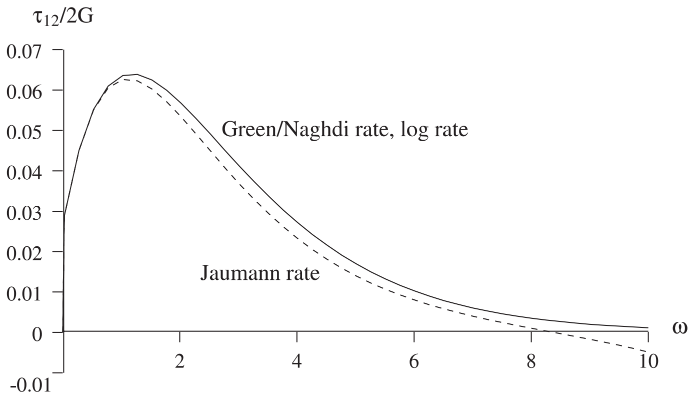

Fig. 3. True shear stresses vs shear strain.

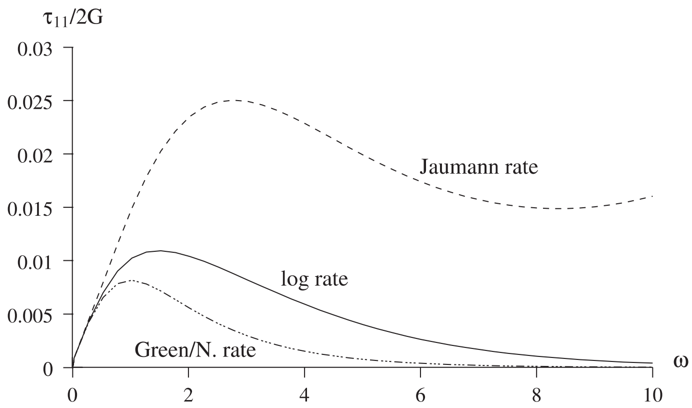

Fig. 4. True normal stresses vs shear strain.

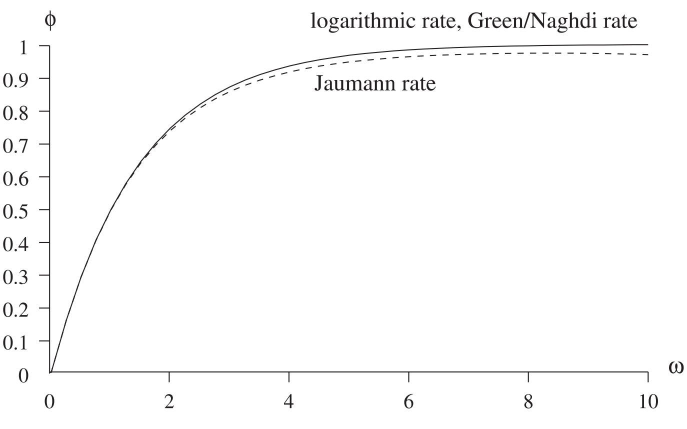

Fig. 5. Void volume fraction vs shear strain.

with a maximum at about $\omega = 1.2$, and asymptotically tending to zero, the result for the Zaremba–Jaumann rate again changes its sign, as for the effective shear stress.

In Fig. 4, the true normal stresses show physically plausible behaviour for both the logarithmic rate and the Green–Naghdi rate, except for the Zaremba–Jaumann rate, which again leads to an oscillating response.

The void volume fraction ϕ vs the shear strain ω is shown in Fig. 5. The results for the Green–Naghdi rate and the logarithmic rate are again almost identical. For very large values of shear strain ω, the void volume fraction ϕ tends asymptotically to the value 1. As has been emphasized by different authors, this indeed will restrict the validity of our fundamental assumptions.

8. Conclusion

It has been demonstrated that, by consistently combining additive and multiplicative decomposition of the stretching $\mathbf{D}$ and the deformation gradient $\mathbf{F}$ and adopting the logarithmic stress rate, a general Eulerian rate type model for finite deformation elastoplasticity coupled with isotropic damage is proposed. The new model is shown to be self-consistent in the sense that the incorporated rate equation for the damaged elastic response is exactly integrable to yield a damaged elastic relation between the effective Kirchhoff stress and the elastic logarithmic strain. The rate form of the new model in a rotating frame in which the foregoing logarithmic rate is defined, is derived and from it an integral form is obtained. The former is found to have the same structure as the counterpart of the small deformation theory and may be appropriate for numerical integration. The latter indicates, in a clear and direct manner, the effect of finite rotation and deformation history on the current stress and the hardening and damage behaviours. Further, it is pointed out that in the foregoing self-consistency sense of formulating the damaged elastic response the suggested model is unique among all objective Eulerian rate type models of its kind with infinitely many objective stress rates to be chosen. In particular, it is indicated that, within the context of the proposed theory, a natural combination of the two widely used decompositions concerning $\mathbf{D}$ and $\mathbf{F}$ can consistently and uniquely determine the elastic and the plastic parts in the two decompositions as well as all their related kinematical quantities.

As an application, the proposed model is applied to derive a self-consistent Eulerian rate type model for void growth and nucleation in metals experiencing finite elastic–plastic deformation by incorporating a modified Gurson's yield function and an associated flow rule, etc. As a test problem, the finite simple shear response of the model is studied by means of numerical integration. It turned out from these calculations that the results obtained for the effective and the true stresses can represent the experimentally observed behaviour, when the logarithmic rate is incorporated in the model. It has been demonstrated further that the Zaremba–Jaumann rate and the Green–Naghdi rate as well may lead to a physically nonplausible behaviour.

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Appendix A

In this appendix, some calculations related with the constitutive model describing the elastic–plastic damage behaviour are comprised.

For the modified Gurson's yield function (Eq. (72)), we have

$$\frac{\partial f}{\partial \boldsymbol{\tau}} = 3(\boldsymbol{\tau}' - \boldsymbol{\alpha}) + q_1 \sigma_F \phi \operatorname{sh}\left(\frac{\operatorname{tr}\boldsymbol{\tau}}{2\sigma_F}\right)\mathbf{I}, \tag{A.1}$$

$$\frac{\partial f}{\partial \boldsymbol{\alpha}} = -3(\boldsymbol{\tau}' - \boldsymbol{\alpha}), \tag{A.2}$$

$$\frac{\partial f}{\partial \phi} = 2q_1\sigma_F^2 \operatorname{ch}\left(\frac{\operatorname{tr}\boldsymbol{\tau}}{2\sigma_F}\right) - 2q_2\sigma_F^2\phi, \tag{A.3}$$

$$\frac{\partial f}{\partial \sigma_F} = 4q_1\sigma_F\phi \operatorname{ch}\left(\frac{\operatorname{tr}\boldsymbol{\tau}}{2\sigma_F}\right) - q_1\phi(\operatorname{tr}\boldsymbol{\tau})\operatorname{sh}\left(\frac{\operatorname{tr}\boldsymbol{\tau}}{2\sigma_F}\right) - 2\sigma_F(1 + q_2\phi^2). \tag{A.4}$$

Introducing these derivatives into Eqs. (79), (81) and (82), yields

$$\dot{\sigma}_F = \dot{\lambda}\frac{\delta}{(1-\phi)\sigma_F}\left(3\boldsymbol{\tau} : (\boldsymbol{\tau}' - \boldsymbol{\alpha}) + q_1\sigma_F(\operatorname{tr}\boldsymbol{\tau})\operatorname{sh}\left(\frac{\operatorname{tr}\boldsymbol{\tau}}{2\sigma_F}\right)\right), \tag{A.5}$$

$$\dot{\phi} = \dot{\lambda}\left(3q_1\sigma_F\phi(1-\phi)\operatorname{sh}\left(\frac{\operatorname{tr}\boldsymbol{\tau}}{2\sigma_F}\right) + \frac{A\delta}{(1-\phi)\sigma_F}\left(3\boldsymbol{\tau} : (\boldsymbol{\tau}' - \boldsymbol{\alpha}) + q_1\sigma_F(\operatorname{tr}\boldsymbol{\tau})\operatorname{sh}\left(\frac{\operatorname{tr}\boldsymbol{\tau}}{2\sigma_F}\right)\right)\right). \tag{A.6}$$

$$\mathbf{D} = \mathbb{D} : \overset{\circ}{\boldsymbol{\tau}}{}^{\log} + \dot{\lambda}\left(3(\boldsymbol{\tau}' - \boldsymbol{\alpha}) + q_1\sigma_F\phi\operatorname{sh}\left(\frac{\operatorname{tr}\boldsymbol{\tau}}{2\sigma_F}\right)\mathbf{I}\right). \tag{A.7}$$

Utilizing the evolution equations (74), (79) and (81), from the general formula (35), we derive an expression for the plastic multiplier $\dot{\lambda}$ as follows:

$$\dot{\lambda} = -\frac{\psi}{\beta}\left(3(\boldsymbol{\tau}' - \boldsymbol{\alpha}) : \overset{\circ}{\boldsymbol{\tau}}{}^{\log} + q_1\phi\sigma_F(\operatorname{tr}\dot{\boldsymbol{\tau}})\operatorname{sh}\left(\frac{\operatorname{tr}\boldsymbol{\tau}}{2\sigma_F}\right)\right) \tag{A.8}$$

with

$$\beta = 3q_1\sigma_F\phi(1-\phi)\operatorname{sh}\left(\frac{\operatorname{tr}\boldsymbol{\tau}}{2\sigma_F}\right)\frac{\partial f}{\partial \phi} + \frac{\delta}{(1-\phi)\sigma_F}\left(A\frac{\partial f}{\partial \phi} + \frac{\partial f}{\partial \sigma_F}\right)\left(\boldsymbol{\tau} : \frac{\partial f}{\partial \boldsymbol{\tau}}\right) - 3c\operatorname{tr}(\boldsymbol{\tau}' - \boldsymbol{\alpha})^2, \tag{A.9}$$

where the loading–unloading indicator ψ is given by Eq. (36). Substituting the derivatives $\partial f/\partial \boldsymbol{\tau}$, $\partial f/\partial \phi$ and $\partial f/\partial \sigma_F$ given by Eqs. (A.1)–(A.4) and making use of

$$2q_1\phi\operatorname{ch}\left(\frac{\operatorname{tr}\boldsymbol{\tau}}{2\sigma_F}\right) - 1 - q_2\phi^2 = -\frac{3}{2}\sigma_F^{-2}\operatorname{tr}(\boldsymbol{\tau}' - \boldsymbol{\alpha})^2, \tag{A.10}$$

one can obtain an explicit expression for β as follows:

$$\begin{aligned}\beta = {} & -3c\operatorname{tr}(\boldsymbol{\tau}' - \boldsymbol{\alpha})^2 - 6q_1\sigma_F^3\phi(1-\phi)\operatorname{sh}\left(\frac{\operatorname{tr}\boldsymbol{\tau}}{2\sigma_F}\right)\left(q_2\phi - q_1\operatorname{ch}\left(\frac{\operatorname{tr}\boldsymbol{\tau}}{2\sigma_F}\right)\right) \\ & - \frac{2\delta}{1-\phi}\left(3\boldsymbol{\tau} : (\boldsymbol{\tau}' - \boldsymbol{\alpha}) + q_1\sigma_F(\operatorname{tr}\boldsymbol{\tau})\operatorname{sh}\left(\frac{\operatorname{tr}\boldsymbol{\tau}}{2\sigma_F}\right)\right) \\ & \times\left(A\sigma_F\left(q_2\phi - q_1\operatorname{ch}\left(\frac{\operatorname{tr}\boldsymbol{\tau}}{2\sigma_F}\right)\right) + q_1\phi\frac{\operatorname{tr}\boldsymbol{\tau}}{2\sigma_F}\operatorname{sh}\left(\frac{\operatorname{tr}\boldsymbol{\tau}}{2\sigma_F}\right) + \frac{3\operatorname{tr}(\boldsymbol{\tau}' - \boldsymbol{\alpha})^2}{2\sigma_F^2}\right).\end{aligned} \tag{A.11}$$

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